On the classical solution to the macroscopic model for in – situ leaching of rare metals.

Anvarbek Meirmanov*

Abstract

Initial boundary value problems describing in-situ leaching of rare metals (uranium, nickel, etc.) with an acid solution are considered. We assume that the solid skeleton is an absolutely rigid body. Next we describe the physical process in the pore space (that is at the microscopic level, where a characteristic size is about 5-20 microns) by the Stokes equations for an incompressible fluid coupled with diffusion-convection equations for acid concentrations and chemical reaction products. Since the solid skeleton changes its geometry during dissolution, the boundary "pore space - solid skeleton" is unknown (free) and the pore space has a special periodic structure. To derive a macroscopic mathematical model (characteristic size meters or tens of meters) for this special periodic structure we use Nguenseng's two-scale convergence method.As a result we prove the existence and uniqueness theorem for the obtained macroscopic mathematical model.

Bibliography: 38 titles.

Key words: Free boundary problems, two-scale convergence, homogenization of periodic structures, fixed point theorem.

MOS subject classification: 35R35, 35M13, 35B27

1 Introduction.

In the proposed manuscript we rigorously derive a macroscopic mathematical model of in – situ leaching of rare metals on the base of homogenization in structures with a special periodicity and prove theorems of the existence and uniqueness of the classical solution to the initial - boundary value problem for the corresponding system of differential equations.

The extraction of rare metals by leaching is very important task of the national economy. Natural deposits of uranium, nickel and other rare metals

*Moscow State University of Civil Engineering, Moscow, Russia
e-mail: anvarbek@list.ru.

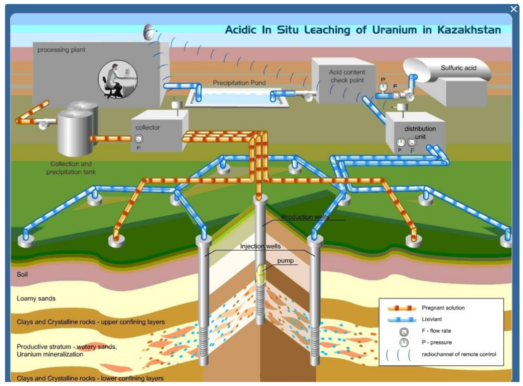

Figure 1: General scheme of the in-situ leaching process

are complex geologically heterogeneous objects. Inhomogeneity means that the properties of an object of interest change in space. Analyses of wells and cores show that the geological properties (porosity, permeability, etc.) of ore bodies are heterogeneous even within a single deposit. Very often insufficient taking into account the consequences of inhomogeneities at the stage of operation planning becomes obvious too late, when the acid solution uploaded to soil through injection wells appeared far from the intended location. In addition, an important role is played by the concentration of the injected acid, the injection modes of acid solutions and other factors.

Hence, understanding the dynamics of fluids in heterogeneous porous media and the mechanism of dissolution of rocks by acids is of fundamental importance for the effective management of rare metal mining. This is achieved by creating a prototype of a hydrodynamic simulator for the ore body based on the corresponding mathematical models, which allows to optimize the entire technological process. A hydrodynamic simulator of an ore body is a complex, consisting of a scale of mathematical models of a physical process (prototype of the hydrodynamic simulator), digital characteristics of the geometric and physical properties of the solid skeleton, and a set of computer programs, that allow one to visualize the physical process and determine the dynamics changes in the main characteristics of the mathematical model.

Currently, a large range of mathematical models exist to describe the dynamics of rock leaching, which may describe the physical processes under consideration, but only at the ***Macroscopic Level*** (see [1] - [3] and references there). Unlike ***Microscopic Model***s (the characteristic size is approximately tens of microns) in ***Macroscopic Model***s, the characteristic size is meters or tens of meters. Because of this, these models do not distinguish between the microstructure of a continuous medium, since in such a model at each point the medium contains both solid skeleton and liquid in the pores or cracks of this skeleton.

All such models are built on the same principle. Fluid dynamics is usually controlled by the Darcy's system of filtration, or some modification of it. The equations describing the migration of acid and chemical reaction products simply postulated and, roughly speaking, are some modifications of diffusion-convection equations for the corresponding concentrations. The main thing in these postulates is the type of coefficients of the equations.

Exactly here there is a great variety of models, depending on the tastes and preferences of its authors. It is quite understandable, since the main mechanism of the physical process is focused on the unknown (free) boundary between the pore space and the solid skeleton and not spelled out in any way in the proposed macroscopic models. This is where the rocks dissolve, changing the concentration of the injected acid, and this is where products appears inside the carrier liquid. Moreover, during the process, the geometry of the pore space (geometry of the boundary separating solid skeleton and pore space) changes in time and space.

These fundamentally important changes occur at the microscopic level, corresponding to the average size of pores or cracks in rocks, while all of proposed macroscopic models operate with completely different orders of scales (tens of centimeters or meters) and, therefore, do not distinguish free boundaries, nor features of the interaction of acid with rocks, which explains the wide variety of macroscopic mathematical models.

The authors of such models simply do not have an exact method for describing physical processes at the microscopic level based on the fundamental laws of the classical continuum mechanics and chemistry, nor the possibility to take into account the microstructure in their macroscopic models. Therefore, they have to restrict themselves with some certain speculative considerations.

Due to this, a natural question arises. If there are several macroscopic models describing the same physical process under the same conditions, which of them most adequately reflects this process? Where is the criterion of the adequacy here? It doesn't make sense to talk about an experiment, since each such model has enough free parameters that are not related to reservoir geometry (e.g. porosity), or to the physical characteristics of the process (such as the viscosity of the filtered liquids, or the physical properties of the solid skeleton and the like). So, with the variation of these free parameters one may get a match with any experiment.

R. Burridge, J. B. Keller [4] and E. Sanchez-Palencia [5] were the first who explained that the exact description a filtration of liquids and seismic in rocks at the macroscopic level is possible if and only if:

(a) the physical process under consideration is described at the microscopic level by equations of Newtonian classical continuum mechanics (exact model);

(b) a set of small dimensionless parameters is selected;

(c) the macroscopic mathematical models is an exact asymptotic limits (homogenization) of exact mathematical models at the microscopic level, when the selected small parameters tend to zero.

Various special cases of exact macroscopic models of acoustics and fluid filtration in rocks have been investigated by many authors: Gilbert-Lin [6], Ferrin, Mikelic [7], Levi [8], Sanchez-Hubert [9], Zhikov, Kozlov, Oleinik [10], Zhikov [11], Pastukhova [12], Bachvalov, Panasenko [13] and others. All these authors used various methods of homogenization and, as a rule, the resolution to each of them was a difficult task required considerable effort and ingenuity.

Everything changed after appearance of G. Nguetseng's paper[14], where the author proposed the ***Method of Two-Scale Convergence*** in periodic structures. What used to be an art has become an ordinary routine, a reference to the method. So the homogenization theory has ceased to be an independent part of the mathematical analysis (or the theory of differential equations) and main efforts in homogenization have moved from theory to applications in mechanics, physics, biology, etc.

Since the problems in periodic structures have been well studied, the attention of researchers began to attract more complex problems in ***Media with special periodicity***, set by characteristic functions of the form $\chi^{\varepsilon}(\boldsymbol{x},t)=\chi(\boldsymbol{x},t,\dfrac{\boldsymbol{x}}{\varepsilon})$, where $\chi(\boldsymbol{x},t,\boldsymbol{y})$ – 1 – periodic by variable $\boldsymbol{y}$ function.

The most fully exact models of physical processes in poro-elastic media were investigated by A. Meirmanov [15]. He rewrote these models in a special dimensionless form and fixed a certain ***Set of criteria*** (dimensionless coefficients of differential equations) that are responsible for the type of physical process (filtration, acoustics, hydraulic shock, etc.). Meirmanov classified accurate mathematical models at the microscopic level and using the Nguetseng multi-scale convergence method obtained exact asymptotic limits of these models that adequately describe the physical processes under consideration at the macroscopic level.

In our manuscript, the value $\varepsilon=\dfrac{l}{L}$, is taken as a small dimensionless parameter. Here l is characteristic pore size and L is the characteristic size of the physical domain under consideration. The dimensionless parameter $\alpha_{\mu}^{\varepsilon}$ characterizes the viscosity of the filtered liquid:

$$\alpha_{\mu}^{\varepsilon}=\frac{2\,\mu}{L\,g\,\tau\,\rho_0},$$

where τ is the characteristic duration time of the physical process, ρ_0 is the density of water, g is the acceleration of gravity and μ is the dynamic viscosity of the liquid.

As we have already noted, the derivation of macroscopic mathematical models should be based on the most accurate mathematical model of the

physical process at the microscopic level, described by the laws of classical continuum mechanics. In our description, these are the Stokes equations in the pore space Ω^{ε}_{f} for an incompressible viscous fluid with dynamic characteristics $\boldsymbol{v}^{\varepsilon}(\boldsymbol{x},t)$(***Velocity vector*** (in the liquid component), $p^{\varepsilon}(\boldsymbol{x},t)$ (***Pressure*** (in the liquid component) and the diffusion-convection equations for ***Concentration acid*** $c^{\varepsilon}(\boldsymbol{x},t)$ and ***Concentrations of chemical reaction products*** $c^{\varepsilon}_{j}(\boldsymbol{x},t)$, $j=1,...,n$.

The solid skeleton Ω^{ε}_{s} is assumed to be **Absolutely rigid** in which the velocities of the medium are zero.

Differential equations are supplemented by boundary conditions at the given boundaries, initial conditions and strong discontinuity conditions on the unknown (**Free**) boundary Γ^{ε} "solid skeleton-pore space" in the domain $\Omega=\Omega^{\varepsilon}_{f}\cup\Gamma^{\varepsilon}\cup\Omega^{\varepsilon}_{s}$ [16],arising from the conservation laws of classical mechanics and an additional condition on the free boundary

$$D^{\varepsilon}_{N}=\alpha^{\varepsilon}\,c^{\varepsilon},\ \ \alpha^{\varepsilon}=\mathrm{const}>0, \tag{1.1}$$

postulated in theoretical chemistry and allows us to find a free boundary [17].

Here D^{ε}_{N} is the velocity of the boundary Γ^{ε} in the direction of the unit normal $\boldsymbol{N}^{\varepsilon}$ to the boundary Γ^{ε}, outward to the domain Ω^{ε}_{f}, α is a given constant.

The formulation of the problem was previously given in the monograph [18], but there were no exact results about the existence of any solution in this monograph.

Note that the physical process we are considering has a rather long duration (the filtration rate of the liquid is several meters per year). Therefore, the most interesting are the theorems of the existence of solutions of the corresponding initial - boundary value problems globally in time. On the other hand, due to the strong nonlinearity of problems with free boundaries [19], it is usually not possible to prove any result globally in time for mathematical models at the microscopic level. That is, possible results can only be theorems of the existence of a generalized or classical solution of the initial - boundary value problem for a system of differential equations describing the leaching process at the macroscopic level.

But how can we obtain macroscopic mathematical models if we know nothing about the existence of solutions to microscopic mathematical models, the limit of which they should be?

To get around these difficulties, we will use the fixed point theorems [20]. To do this, the structure of the pore space is fixed, given by the characteristic function $\chi(\boldsymbol{x},t,\boldsymbol{y})$. As we have already noted, in the case of a general provision,to solve the emerging problem is almost impossible. Therefore, it is reasonable to limit ourselves to the simplest cases. For example, when a non-negative function $r(\boldsymbol{x},t)$ from some set $\mathfrak{M}_{T}$ defines a characteristic function of the pore space $\chi(r,\boldsymbol{y})$. Next, for a fixed $r\in\mathfrak{M}_{T}$, the initial boundary

value problem $\mathbb{B}^\varepsilon(r)$ is considered in a given domain $\Omega^\varepsilon_f(r)$ for determining the main characteristics of the medium (velocity vector, pressure and acid concentration), which is the initial boundary value problem $\mathbb{A}^\varepsilon$ without an additional boundary condition (1.1). In this auxiliary problem, for a fixed $\varepsilon > 0$, the solid skeleton is a union of disjoint sets sufficiently close to balls of radius $\varepsilon\, r$, slowly decreasing in volume, which simplifies the geometry of the original pore space and allows us to prove the existence of approximate solutions.

To understand what **s**hould be theHomogenized Problem $\mathbb{H}(r)$ of the problems $\mathbb{B}^\varepsilon(r)$, a formal homogenization of the problem $\mathbb{A}^\varepsilon$ is performed beforehand. The conditions for the existence of such homogenization are formulated in the lemma 4.1.

If $r^\varepsilon(\boldsymbol{x}, t)$ defines the structure of the solid skeleton (pore space) in the problem $\mathbb{A}^\varepsilon$ and $r_\varepsilon \to r^*$ at $\varepsilon \to 0$, then homogenize problem $\mathbb{H}(r^*)$ of the problem $\mathbb{B}^\varepsilon(r^*) = \mathbb{A}^\varepsilon$ should coincide with the homogenization $\mathbb{H}$ of the problems $\mathbb{A}^\varepsilon$ without homogenization of the boundary condition (1.1). It is clear that the homogenization of the boundary condition (1.1) with a given pore space structure $r(\boldsymbol{x}, t)$ forms a problem operator whose only fixed point $r^*(\boldsymbol{x}, t)$ determines the required unique homogenization $\mathbb{H}$ of the problem $\mathbb{A}^\varepsilon$.

To solve the problem $\mathbb{H}(r)$ we, first of all, have to solve the problem $\mathbb{B}^\varepsilon(r)$ and then find its homogenization $\mathbb{H}(r)$ for $\varepsilon \to 0$.

The linear problem $\mathbb{B}^\varepsilon(r)$decomposes into a sequential solution **Dynamic problem** $\mathbb{B}^\varepsilon(r)$ on the definition of dynamic characteristics $\boldsymbol{v}^\varepsilon(\boldsymbol{x}, t)$ and $p^\varepsilon(\boldsymbol{x}, t)$ and **Diffusion-convection problems** $\mathbb{B}^\varepsilon(r)$ for finding the acid concentration $c^\varepsilon(\boldsymbol{x}, t)$ and concentrations of products of chemical reactions $c^\varepsilon_j(\boldsymbol{x}, t)$, $j = 1, ..., n$.

Due to the linearity of these problems, the existence and uniqueness of a weak solution to each of them follows from the corresponding a priori estimates and known methods for solving linear differential equations. For example, the ***Galerkin's method*** [21].

The next step is to homogenization the resulting mathematical model $\mathbb{B}^\varepsilon(r)$. To homogenize this problem, we use the Nguetseng's two-scale convergence method [14], which makes it quite easy to get a mathematical model $\mathbb{H}(r)$. But since this method was developed only for averaging functionals, we needed to write down the original mathematical model in the form of a system of integral identities equivalent to the original system of differential equations.

If the integral identities equivalent to the dynamic Stokes equations, as well as the integral identities for the diffusion-convection equation with classical boundary conditions, are well known, then a similar identity for acid concentration, including the diffusion-convection equation and boundary conditions on the free boundary, has been unknown until now.

Equivalent expression of differential equations in the form of integral iden-

tities is a common and rather difficult task for free boundary problems. The happy exception was the Stefan problem [22], [23], describing phase transitions in pure (without impurities) media. Such, for example, as "water – ice" or chemically pure metals. The authors of the cited works O. A. Oleinik and S. L. Kamenomostskaya (S. Kamin) managed to reformulate the problem so that the equivalent formulation in the form of an integral identity, in the case of the existence of a classical solution, contained both the equation of thermal conductivity outside the free boundary and the condition itself on the free boundary. This approach allowed them to prove quite simply the existence and uniqueness of the weak solution to the integral identity under minimal conditions for the smoothness of the solution. At the same time, in the case of the existence of a classical solution of the Stefan problem, the latter must coincide with the weak solution. The question of the existence of a classical solution to the one-phase Stefan problem remained open until 1975 [24]. The existence of a classical solution to the two-phase Stefan problem was proved in 1979 [19].

Considering the above, in the mathematical model of in – situ leaching proposed by us, it was first of all very important to find such an equivalent formulation of the problem in the form of a system of integral identities that would require minimal smoothness of solutions to the problem, which was done.

At the same time, for the fluid velocity $\boldsymbol{v}^{\varepsilon}$ and the acid concentration c^{ε}, defined only in pore space $\Omega^{\varepsilon}_{f,T}(r)=\bigcup\limits_{t=0}^{t=T}\Omega^{\varepsilon}_{f}(r)$, it was necessary to find an extension of these solutions from the domain of definition onto domain $\Omega_T=\Omega\times(0,T)$ while preserving their best differential properties. To do this, we used the results on the extension of functions, formulated in Lemmas 2.8 and 2.9 § 2, so that the extensions $\widetilde{\boldsymbol{v}}^{\varepsilon}(\boldsymbol{x},t)$, $\widetilde{p}^{\varepsilon}(\boldsymbol{x},t)$ and $\widetilde{c}^{\varepsilon}(\boldsymbol{x},t)$ of functions $\boldsymbol{v}^{\varepsilon}(\boldsymbol{x},t)$, $p^{\varepsilon}(\boldsymbol{x},t)$ and $c^{\varepsilon}(\boldsymbol{x},t)$ satisfy the integral identities of conservation of momentum, mass and acid concentration, equivalent to the corresponding differential equations together with the boundary and initial conditions.

The moving boundary $\Gamma^{\varepsilon}(r)$ creates the main problem when deriving a'priori estimates to solutions to the problem $\mathbb{B}^{\varepsilon}(r)$. Standard methods to getting a priori estimates by multiplying a differential equation by solutions (or a combination thereof) of these equations in free boundary problems often do not lead to the desired result. In our situation, this method helped only partially.

Similarly, the derivation of a priori estimates of solutions to the diffusion – convection problem $\mathbb{B}^{\varepsilon}(r)$, as a rule, is based on the principle of the maximum inherent in standard initial - boundary value problems for parabolic equations. Unfortunately, there is no maximum principle in the initial formulation of our problem. Unfortunately, there is no maximum principle in

the initial formulation of the problem.

All of the above required the introduction of the **Modified diffusion – convection problem** $\mathbb{B}^{\varepsilon}(r)$,which coincides with diffusion – convection problem $\mathbb{B}^{\varepsilon}(r)$, if the concentration of acid in modified diffusion – convection problem $\mathbb{B}^{\varepsilon}(r)$ satisfies the constraints of the standard maximum principle (5.5).

A'priori estimates of weak solutions (that is, solutions to the corresponding integral identities), as a rule need a special selection of trial functions in the integral identity and integration in parts. For the latter, sufficient smoothness of the boundary of the pore space $\Omega^{\varepsilon}_{f}(r)$ (the domain filled with liquid) is necessary. The smoothness of the boundary $\partial\Omega^{\varepsilon}_{f}(r)$ is defined by $r \in \mathfrak{M}_T$. This simple fact is central one to the derivation of a'priori estimates.

Let's briefly focus on the structure of the article.

In § 2 well-known facts and definitions are given, as well as new results on the compactness of sequences in a non-periodic structure (Theorem 2) and results on the extension of functions defined in a pore space $\Omega^{\varepsilon}_{f,T}(r)$ with a given structure, defined by the function $r(\boldsymbol{x},t)$ are proved (Lemmas 1.2.8 and 1.2.9).

§ 3 is devoted to the formulation of an initial boundary value problem describing a physical process at the microscopic level.

In § 4 the formal homogenization of the problems $\mathbb{A}^{\varepsilon}$ and $\mathbb{B}^{\varepsilon}(r)$ are considered.

To get integral identities equivalent to differential equations supplemented with the corresponding boundary and initial conditions we assume the necessary smoothness of the solutions to the original problem. These integral identities allow us to correctly define the weak solutions to problems $\mathbb{A}^{\varepsilon}$ and $\mathbb{B}^{\varepsilon}(r)$.

In Lemmas 4.1 –4.3 we formulate the necessary conditions for homogenization the problems $\mathbb{A}^{\varepsilon}$ and $\mathbb{B}^{\varepsilon}(r)$ and formally derive homogenized models $\mathbb{H}(r)$ and $\mathbb{H}$ or the limiting velocity, pressure and acid concentration.

In Lemma 4.2 under assumption (4.18) we derive integral identity (4.20), equivalent to the boundary condition (1.1), and find the homogenization (4.19) of this boundary condition. On the base of equality (4.19) we construct the problem operator

$$\mathbb{F}(r) = -\theta \int_0^t c(\boldsymbol{x},\tau)d\tau.$$

The fixed point of this operator define the desired structure of the pore space and auxiliary operators $\mathbb{F}^c(r) = c(\boldsymbol{x},t)$, $\mathbb{F}^v(r) = \boldsymbol{v}(\boldsymbol{x},t)$ $\mathbb{F}^p(r) = p(\boldsymbol{x},t)$, where $\{\boldsymbol{v},\, p,\, c\}$ are solutions to the problem $\mathbb{H}(r)$.

The constant θ is defined by equality (4.18).

In Lemma 4.3 we prove the correct definition of these operators $\mathbb{F}^c(r)$, $\mathbb{F}^v(r)$ and $\mathbb{F}^p(r)$.

In § 5 we formulate the main results of the manuscript.

In § 6 we prove Theorem 3. The crucial moment here is a'priory estimates for the problem $\mathbb{B}^{\varepsilon}(r)$.

In § 7 we prove Theorem 4 on existence of weak and classical solutions to the problem $\mathbb{H}(r)$.

§ 8 is devoted to the prove of Theorem 5. In this theorem we analyze properties of opderators $\mathbb{F}^c(r)$, $\mathbb{F}^v(r)$ and $\mathbb{F}^p(r)$and the principal operator $\mathbb{F}(r)$.

We show that operator $\mathbb{F}(r)$ is a Lipschitz continuous one with the corresponding constant bounded by some linear function of T. This fact proves the existence of the unique fixed point $r^*(\boldsymbol{x},t)$ locally in time.

The last statement means the uniqueness of the solutions to the mathematical model $\mathbb{H}$.

Finally, using smoothness of the solutions to the problem $\mathbb{H}(r)$ we prove the correctness of the mathematical model $\mathbb{H}$ for any $T>0$.

In our paper, we used the notation adopted in [25]and in [26].

2 Auxiliary statements.

2.1 Notations.

2.1.1 Dimensionless parameters.

We have already defined dimensionless parameters α_μ c_f in the introduction.

Diffusion of acid and chemical reaction products is characterized by dimensionless diffusion coefficients

$$\alpha_c=\frac{D\,T}{L^2}=d_0,\ \ \alpha_i=\frac{D_i\,T}{L^2},\ \ i=1,...,n,$$

respectively, acids and products of chemical reactions.

We will assume that α_μ depends on the small parameter ε: $\alpha_\mu=\alpha_\mu^\varepsilon$, and there are finite or infinite limits

$$\lim_{\varepsilon\to 0}\alpha_\mu^\varepsilon=\mu_0,\ \ \lim_{\varepsilon\to 0}\frac{\alpha_\mu^\varepsilon}{\varepsilon^2}=\mu_1.$$

The coefficients c_f, d_0 α_i are fixed and do not depend on the small parameter ε.

As ϱ_s we denote the dimensionless density of the solid skeleton, related to the density of water ρ_0 and as ϱ_f – dimensionless density of the liquid component related to the density of water ρ_0.

2.1.2 Domains and boundaries.

$\Omega\subset\mathbb{R}^3$ is a bounded domain with peace piecewise smooth boundary S, $\bar{S}=\partial\Omega$, $S=S^0\cup S^1\cup S^2$, $S^i\cap S^0=\emptyset$, $i=1,2$, $S^1\cap S^2=\emptyset$, $S^j_T=S^j\times(0,T)\subset\mathbb{R}^4$, $j=0,1,2,3$, $\Omega_T=\Omega\times(0,T)\subset\mathbb{R}^4$.

The boundary $S^0 \subset \mathbb{R}^3$ is impermeable to liquid in the pore space,, the boundary $S^1 \subset \mathbb{R}^3$ simulates injection wells and boundary $S^2 \subset \mathbb{R}^3$ simulates production wells.

For simplicity of presentation, in what follows we assume Ω the unit cube, $S^0 = \{\boldsymbol{x} : x_3 = \pm\frac{1}{2}, |x_1| < \frac{1}{2}, |x_2| < \frac{1}{2}\}, \cup\{\boldsymbol{x} : x_2 = \pm\frac{1}{2}, |x_1| < \frac{1}{2}, |x_3| < \frac{1}{2}\}$, $S^1 = \{\boldsymbol{x} : x_1 = \frac{1}{2}, |x_2| < \frac{1}{2}, |x_3| < \frac{1}{2}\}$, $S^2 = \{\boldsymbol{x} : x_1 = -\frac{1}{2}, |x_2| < \frac{1}{2}, |x_3| < \frac{1}{2}\}$.

Let $r = r(\boldsymbol{x}, t)$, $0 \leqslant r(\boldsymbol{x}, t) \leqslant \frac{1}{2}$, $(\boldsymbol{x}, t) \in \Omega_T$, be a given function, that determines the structure of the pore space $\Omega^\varepsilon_f(r) \subset \Omega$ and the solid skeleton $\Omega^\varepsilon_s(r) \subset \Omega$ in Ω and the structure of the pore space $\Omega^\varepsilon_{f,T}(r) = \bigcup\limits_{t=0}^{t=T} \Omega^\varepsilon_f(r) \subset \Omega_T$ and the solid skeleton $\Omega^\varepsilon_{s,T}(r) = \bigcup\limits_{t=0}^{t=T} \Omega^\varepsilon_s(r) \subset \Omega_T$ in Ω_T.

Let also the boundary $\Gamma^\varepsilon(r) = \partial\Omega^\varepsilon_f(r) \bigcap \partial\Omega^\varepsilon_s(r)$ divides liquid and solid components in Ω and $\Gamma^\varepsilon_T(r) = \bigcup\limits_{t=0}^{t=T} \Gamma^\varepsilon(r)$ in Ω_T.

2.1.3 Structure of pore space.

In what follows all functions of the type $\Phi(\boldsymbol{x}, t, \boldsymbol{y})$, where $(\boldsymbol{x}, t) \in \Omega$ $\boldsymbol{y} \in \mathbb{R}^3$ are considered 1 - periodic in variable $\boldsymbol{y}$:

$$\Phi(\boldsymbol{x}, t, \boldsymbol{y}) = \Phi\big(\boldsymbol{x}, t, \boldsymbol{\varsigma}(\boldsymbol{y})\big), \ \boldsymbol{y} = [|\boldsymbol{y}|] + \boldsymbol{\varsigma}(\boldsymbol{y}), \ [|\boldsymbol{y}|] = ([|y_1|], [|y_2|], [|y_3|]). \tag{2.1}$$

Here $[|a|]$ is the integer part of the number a.

For fixed $\varepsilon > 0$ the pore space $\Omega^\varepsilon_f(r)$ and the solid skeleton $\Omega^\varepsilon_s(r)$ are defined by 1 – periodic in variable $\boldsymbol{y}$ characteristic function $\chi(r, \boldsymbol{y})$ as:

$$\begin{gathered}\Omega^\varepsilon_f(r) = \mathrm{Int}\{\boldsymbol{x} \in \Omega : \chi^\varepsilon(\boldsymbol{x}, t) = 1\}, \\ \Omega^\varepsilon_s(r) = \mathrm{Int}\{\boldsymbol{x} \in \Omega : \chi^\varepsilon(\boldsymbol{x}, t) = 0\}, \ \Omega^\varepsilon_f(0) = \emptyset, \ \Omega^\varepsilon_s(0) = \Omega; \\ \chi^\varepsilon(\boldsymbol{x}, t) = \chi\big(r(\boldsymbol{x}, t), \frac{\boldsymbol{x}}{\varepsilon}\big), \ \chi(r, \boldsymbol{y}) = \frac{\mathrm{sgn}\big(r(\boldsymbol{x}, t) - |\boldsymbol{\varsigma}(\boldsymbol{y})|\big) + 1}{2}. \end{gathered} \tag{2.2}$$

Let

$$\overline{\Omega}=\bigcup_{\boldsymbol{k}\in\mathbb{Z}}\overline{\Omega}^{\boldsymbol{k},\varepsilon},\;\;\Omega^{\boldsymbol{k},\varepsilon}=\{\boldsymbol{x}\in\Omega:\boldsymbol{x}=\varepsilon\boldsymbol{k}+\varepsilon\boldsymbol{y}\},\;\;\Omega_T^{\boldsymbol{k},\varepsilon}=\Omega^{\boldsymbol{k},\varepsilon}\times(0,T),$$

$$\Omega_f^{\boldsymbol{k},\varepsilon}(r)=\Omega_f^{\varepsilon}(r)\cap\Omega^{\boldsymbol{k},\varepsilon},\;\;\Omega_s^{\boldsymbol{k},\varepsilon}(r)=\Omega_s^{\varepsilon}(r)\cap\Omega^{\boldsymbol{k},\varepsilon},$$

$$\Omega_{f,T}^{\boldsymbol{k},\varepsilon}(r)=\bigcup_{t=0}^{T}\Omega_f^{\boldsymbol{k},\varepsilon}(r),\;\;\Omega_{s,T}^{\boldsymbol{k},\varepsilon}(r)=\bigcup_{t=0}^{T}\Omega_s^{\boldsymbol{k},\varepsilon}(r).$$

$$\text{If } r(\varepsilon\,\boldsymbol{k},t)=0,\;\;\Omega_f^{\varepsilon}(r)=\emptyset\;\;\Omega_s^{\boldsymbol{k},\varepsilon}(r)=\Omega^{\boldsymbol{k},\varepsilon}$$

for all $\boldsymbol{k}=(k_1,k_2,k_3)$, $k_1,k_2,k_3\in\mathbb{Z}$ (integer numbers) and for all $\boldsymbol{y}=(y_1,y_2,y_3)\in Y$.

Here and in what follows $\dfrac{1}{\varepsilon}$ is an integer. For this case $\Omega^{\boldsymbol{k},\boldsymbol{\varepsilon}}\subset\Omega$ for all $\boldsymbol{k}$.

According to this decomposition, a free boundary $\Gamma^{\varepsilon}(r)$ is represented in Ω as the following set:

$$\Gamma^{\varepsilon}(r)=\{\boldsymbol{x}\in\Omega:|\varsigma(\frac{\boldsymbol{x}}{\varepsilon})|=r\}=\bigcup_{\boldsymbol{k}\in\mathbb{Z}}\Gamma^{\boldsymbol{k},\varepsilon}(r)\subset\Omega,\;\;\Gamma_T^{\varepsilon}(r)=\bigcup_{t=0}^{t=T}\Gamma^{\varepsilon}(r)\subset\Omega_T,$$

$$\Gamma^{\boldsymbol{k},\varepsilon}(r)=\Gamma^{\varepsilon}(r)\bigcap\Omega^{\boldsymbol{k},\varepsilon},\;\;\Gamma^{\boldsymbol{k},\varepsilon}(r)=\{\boldsymbol{x}\in\Omega^{\boldsymbol{k},\varepsilon}:|\varsigma(\frac{\boldsymbol{x}}{\varepsilon})|=r(\boldsymbol{x},t)\}.$$

The structures of the pore space and the solid skeleton are characterized by elementary cells $Y_f(r)$ $Y_s(r)$:

$$Y_f(r)=\mathrm{Int}\{\boldsymbol{y}\in Y:\chi(r,\boldsymbol{y})=1\},\;\;Y_s(r)=\mathrm{Int}\{\boldsymbol{y}\in Y:\chi(r,\boldsymbol{y})=0\},$$

where $Y=(-\frac{1}{2},\frac{1}{2})^3\subset\mathbb{R}^3$

To the movement with increasing time of each component $\Gamma^{\varepsilon,\boldsymbol{k}}(r)$ of the boundary $\Gamma^{\varepsilon}(r)$ in Ω corresponds the movement of the boundary $\gamma(r)$ in Y: $\gamma(r)=\{\boldsymbol{y}\in Y:\,|\boldsymbol{y}|=r\}$.

As $d_n(r)$ in variables $(\boldsymbol{y},t)$ we denote the velocity of the boundary $\gamma(r)$ in Y in the direction of the normal $\boldsymbol{n}$ to $\gamma(r)$ in the pount $\boldsymbol{y}\in\gamma(r)$, outward to $Y_f(r)$:

$$d_n(r)=-\frac{\partial r}{\partial t}(\boldsymbol{x},t).\tag{2.3}$$

Similarly, as $D_N^{\varepsilon}(r)$ we denote the velocity of the free boundary $\Gamma^{\varepsilon}(r)$ in the direction of the normal $\boldsymbol{N}^{\varepsilon}$ to $\Gamma^{\varepsilon}(r)$ at the point $\boldsymbol{x}\in\Gamma^{\varepsilon}(r)$, outward to $\Omega_f^{\varepsilon}(r)$.

Finally, for functions Ψ of the type $\Psi=\Psi(\boldsymbol{x},t,\boldsymbol{y})$ we define average by period

$$\langle\Psi\rangle_Y=\int_Y\Psi\,dy,\;\;\langle\Psi\rangle_{Y_f(r)}=\int_Y\chi(r,\boldsymbol{y})\Psi\,dy,\;\;\langle\Psi\rangle_{Y_s(r)}=\int_Y\big(1-\chi(r,\boldsymbol{y})\big)\Psi\,dy.$$

In particular,for given structure $r(\boldsymbol{x},t)$ with characteristic function $\chi(r,\boldsymbol{y})$, given by formula (2.1), the function

$$m(r) = \int_Y \chi(r,\boldsymbol{y})dy = \langle\chi\rangle_Y = 1 - \frac{4}{3}\pi r^3 \geqslant \frac{1}{3},$$

is the **Porosity** of the solid skeleton in the point $(\boldsymbol{x},t)$.

2.1.4 Differential operators and matrices.

Operator ∇ without indices means differentiation with respect to the variable $\boldsymbol{x}$;

Operator ∇_y means differentiation with respect to the variable $\boldsymbol{y}$.

$\mathbb{D}(x,\boldsymbol{u}) = \frac{1}{2}\big(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^*\big)$.

$\mathbb{I}$ is a unit matrix.

$\mathbb{B}:\mathbb{C} = tr(\mathbb{B}\cdot\mathbb{C}^*)$, where $\mathbb{B}$, $\mathbb{C}$ are **Second-Order Tensors** (matrices).

$\boldsymbol{a}\otimes\boldsymbol{b}$ is **Dyad**, for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$: $(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$.

$\mathbb{J}^{ij} = \frac{1}{2}(\boldsymbol{e}^i\otimes\boldsymbol{e}^j + \boldsymbol{e}^j\otimes\boldsymbol{e}^i)$, $\{\boldsymbol{e}^1,\ \boldsymbol{e}^2,\ \boldsymbol{e}^3\}$ is the standard Cartesian basis in $\mathbb{R}^3$.

$\mathbb{A}\otimes\mathbb{B}$ is **Forth-Order Tensor**.

$(\mathbb{A}\otimes\mathbb{B}):\mathbb{C} = \mathbb{A}(\mathbb{B}:\mathbb{C})$ for any second order tensors $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$.

2.2 Two – scale convergence [14].

In the present section we consider 1 – periodic in the variable $\boldsymbol{y}\in Y=(0,1)^3$ functions $W(\boldsymbol{x},t,\boldsymbol{y})$ with $(\boldsymbol{x},t)\in\Omega_T$.

Definition 1. *The sequence* $\{w^\varepsilon\}\subset\mathbb{L}_2(\Omega_T)$, *is said to be two – scale convergent to the function* $W(\boldsymbol{x},t,\boldsymbol{y})\in\mathbb{L}_2(\Omega_T\times Y)$, *which is 1 – periodic in the variable* $\boldsymbol{y}\in Y$ *(notation* $w^\varepsilon \stackrel{2-sc.}{\to} W(\boldsymbol{x},t,\boldsymbol{y})$*), if for any smooth function* $\sigma=\sigma(\boldsymbol{x},t,\boldsymbol{y})$, *1 – periodic in the variable* $\boldsymbol{y}$ *is valid the equality*

$$\lim_{\varepsilon\to 0}\int\int_{\Omega_T} w^\varepsilon(\boldsymbol{x},t)\sigma(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})dx\,dt = \int\int_{\Omega_T}\Big(\int_Y W(\boldsymbol{x},t,\boldsymbol{y})\sigma(\boldsymbol{x},t,\boldsymbol{y})dy\Big)dx\,dt. \quad (2.4)$$

The existence and basic properties of two – scale convergent sequences are proved in the following theorem:

Theorem 1. *(Nguetseng's Teorem)*

1. *Any bounded in* $\mathbb{L}_2(\Omega_T)$ *sequence* $\{\boldsymbol{w}^\varepsilon\}$ *contains some subsequence two – scale convergent to some function*
$\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}\in\mathbb{L}_2(\Omega_T\times Y)$, *1 – periodic in the variable* $\boldsymbol{y}$.

2. *Let sequences* $\{\boldsymbol{w}^{\varepsilon}\}$ *and* $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ *are uniformly bounded in* $\mathbb{L}_2(\Omega_T)$.

Then there exists the function $\boldsymbol{W}=\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, *1 – periodic in* $\boldsymbol{y}$, *and the sequence* $\{\boldsymbol{w}^{\varepsilon}\}$ *such that* $\boldsymbol{W}$, $\nabla_y\boldsymbol{W}\in\mathbb{L}_2(\Omega_T\times Y)$, *and sequences* $\{\boldsymbol{w}^{\varepsilon}\}$ *and* $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ *(for simplicity we keep the same indices for subsequences) two – scale converge in* $\mathbb{L}_2(\Omega_T)$ *to* $\boldsymbol{W}$ *and* $\mathbb{D}(y,\boldsymbol{W})$ *correspondingly.*

3. *Let sequences* $\{\boldsymbol{w}^{\varepsilon}\}$ *and* $\{D(x,\boldsymbol{w}^{\varepsilon})\}$ *are bounded in* $\mathbb{L}_2(\Omega_T)$.

Then there exist functions $\boldsymbol{w}(\boldsymbol{x},t)$, $\boldsymbol{w}\in\mathbb{W}^{1,0}_2(\Omega_T)$ *and* $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}\in\mathbb{L}_2(\Omega_T\times Y)\cap\mathbb{W}^{1,0}_2(Y)$, *subsequence from* $\{\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ *such that the function* $\boldsymbol{W}$ *is 1 – periodic in* $\boldsymbol{y}$, $\mathbb{D}(x,\boldsymbol{w})\in\mathbb{L}_2(\Omega_T)$, $D(y,\boldsymbol{W})\in\mathbb{L}_2(\Omega_T\times Y)$, *and the sequence* $\{\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ *two – scale converges to the function* $\mathbb{D}(x,\boldsymbol{w})+D(y,\boldsymbol{W})$.

Consequence 2.1. (Lemma B 13, Appendix B, [15])
If $\alpha(\varepsilon)\|\nabla\boldsymbol{w}^{\varepsilon}\|_{2,\Omega_T}\leqslant M$, *where* M *does not depend on* ε *and* $\lim\limits_{\varepsilon\to 0}\dfrac{\alpha(\varepsilon)}{\varepsilon}=\infty$, *then* $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}(\boldsymbol{x},t)$.

Note, that weak and two – scale convergence are connected by the relation:

$$w^{\varepsilon}\overset{t.-s.}{\to}W(\boldsymbol{x},t,\boldsymbol{y})\Rightarrow w^{\varepsilon}(\boldsymbol{x})\rightharpoonup\int_Y W(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}\ \text{(weakly converges)}.$$

2.3 Solenoidal functions in pore space.

Let $\overset{\circ}{Y}_f\big(r(\boldsymbol{x},t)\big)=Int\bigcap\limits_{(\boldsymbol{x},t)\in\Omega_T}Y_f\big(r(\boldsymbol{x},t)\big)$.

By construction $\overset{\circ}{Y}_f\neq\emptyset$ because $\emptyset\neq Y_f(\frac{1}{2})\subset Y_f(r)$ for all $r(\boldsymbol{x},t),\ 0\leqslant r(\boldsymbol{x},t)\leqslant\dfrac{1}{2}$.

Recall, that some smooth function $\boldsymbol{\varphi}(\boldsymbol{y})$ is called *Solenoidal* in $\overset{\circ}{Y}_f$, $\boldsymbol{y}\in\overset{\circ}{Y}_f$.

We need the following statement, proved in [15] (Appendix B, Lemma B. 15):

Lemma 2.1. *For any unit vector* $\mathbf{e}$ *there exists a solenoidal function* $\boldsymbol{\varphi}(\boldsymbol{y})$ *such that* $\boldsymbol{\varphi}\in\overset{\circ}{\mathbb{W}}{}^{1,0}_2(\overset{\circ}{Y}_f)$, *supp* $\boldsymbol{\varphi}\subset\overset{\circ}{Y}_f\subset Y_f(r)$ *and*

$$\int_Y\boldsymbol{\varphi}(\boldsymbol{y})dy=\boldsymbol{e}. \tag{2.5}$$

2.4 Strong convergence criteria in $\mathbb{L}_2(\Omega_T)$.

Definition 2. We say that the function $u(\boldsymbol{x},t)$, bounded in $\mathbb{L}_2(\Omega_T)$, possesses the time derivative $\dfrac{\partial u}{\partial t}\in\mathbb{L}_2\big(0,T;\mathbb{W}^{-1}_2(\Omega)\big)$, if

$$|\int\int_{\Omega_T}u\,\frac{\partial\xi}{\partial t}dxdt|\leqslant M_u\,|\int\int_{\Omega_T}|\nabla\xi|^2dxdt|^{\frac{1}{2}}$$

for all functions $\xi \in \mathbb{W}_2^{1,1}(\Omega_T)$ with some positive constant M_u independent of ξ.

Lemma 2.2. *(Lions [26], Aubin [27]) Let sequences $\{u^\varepsilon\}$ and $\{\nabla u^\varepsilon\}$ are uniformly bounded in the space $\mathbb{L}_2(\Omega_T)$, and the sequence of derivatives $\{\dfrac{\partial u^\varepsilon}{\partial t}\}$ are uniformly bounded in the space $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$.*

Then there exists some subsequence of the sequence $\{u^\varepsilon\}$ strongly convergent in $\mathbb{L}_2(\Omega_T)$.

The generalization of this lemma for periodic structure with characteristic function $\chi^\varepsilon(\boldsymbol{x}) = \chi(\dfrac{\boldsymbol{x}}{\varepsilon})$ has been proved by Meirmanov, Zimin [28]:

Lemma 2.3. *Let $\chi^\varepsilon(\boldsymbol{x}) = \chi(\dfrac{\boldsymbol{x}}{\varepsilon})$, where $\chi(\boldsymbol{y})$ is 1 – periodic in $\boldsymbol{y}$ function, the sequences $\{c^\varepsilon\}$ and $\{\nabla c^\varepsilon\}$ are uniformly bounded in $\mathbb{L}_2(\Omega_T)$, and the sequence $\{\chi^\varepsilon \dfrac{\partial c^\varepsilon}{\partial t}\}$ is uniformly bounded in $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$.*

Then there exists some subsequence of $\{c^\varepsilon\}$ that converges strongly in $\mathbb{L}_2(\Omega_T)$.

Remark 1. We denote the norm of an element φ in $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$ as $\|\varphi\|_{W_2^{-1}}$.

In this section, we prove a similar result for periodic structures with a special pore space structure.

Theorem 2. *Let the structure $\chi(r,\boldsymbol{y})$ of the pore space be given by (2.1), where $r \in \mathfrak{M}_T$ and*

$$\mathfrak{M}_T = \{r \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T),\ 0 \leqslant r(\boldsymbol{x},t) \leqslant \frac{1}{2},\ -\theta \leqslant \frac{\partial r}{\partial t}(\boldsymbol{x},t) \leqslant 0,$$
$$0<\gamma<1,\ \ \theta = const > 0;\ \ |r|_{\Omega_T}^{(2+\gamma)} \leqslant M_0\}.$$

Then any sequence $\{\widetilde{c}^\varepsilon\}$, such that

$$\|\widetilde{c}^\varepsilon\|_{2,\Omega_T} + \|\nabla \widetilde{c}^\varepsilon\|_{2,\Omega_T} + \|\frac{\partial \widetilde{c}^\varepsilon}{\partial t}\|_{W_2^{-1}} \leqslant M,$$

where M does not depend on ε, contains a subsequence, strongly convergent in $\mathbb{L}_2(\Omega_T)$.

Note that the non-positiveness of the time derivative of the function $r \in \mathfrak{M}_T$ means non-positiveness of the normal velocity of the boundary $\Gamma^\varepsilon(r)$ in the direction of the normal $\boldsymbol{N}$ to this boundary, outward to the domain $\Omega_f^\varepsilon(r)$.

We divide the proof of the theorem into several steps.

Lemma 2.4. *In conditions of Theorem 2 for almost all $t_0 \in (0,T)$ the sequence $\{\chi^{\varepsilon}(\boldsymbol{x},t_0)\,\widetilde{c}^{\varepsilon}(\boldsymbol{x},t_0)\}$ converges weakly in $\mathbb{L}_2(\Omega)$ to the function $m(\boldsymbol{x},t_0)\,c(\boldsymbol{x},t_0)$.*

Proof. According to Theorem 1 the sequence $\{\widetilde{c}^{\varepsilon}\}$ (up to some subsequence) two – scale converges to some function $c(\boldsymbol{x},t)$ in $\mathbb{L}_2(\Omega_T)$:

$$\lim_{\varepsilon\to 0}\int\int_{\Omega_T}\widetilde{c}^{\varepsilon}(\boldsymbol{x},t)\,\varphi(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})dxdt=\int\int_{\Omega_T}c\,\langle\varphi\rangle_Y dxdt \tag{2.6}$$

for any test functions $\varphi\in\mathbb{L}_2(\Omega_T\times Y)$.

By definition of the time derivative of the function $\widetilde{c}^{\varepsilon}$ from the space $\mathbb{L}_2\big((0,T);\mathbb{W}_2^{-1}(\Omega)\big)$ there exists some sequence of vector – functions $\{\boldsymbol{u}^{\varepsilon}\}$ such that $\|\boldsymbol{u}^{\varepsilon}\|^2_{2,\Omega_T}\leqslant M^2$ and

$$\int\int_{\Omega_T}\widetilde{c}^{\varepsilon}(\boldsymbol{x},t)\,\chi^{\varepsilon}(\boldsymbol{x},t)\,\frac{\partial\varphi}{\partial t}+\boldsymbol{u}^{\varepsilon}\cdot\nabla\varphi\big)\,dxdt=0 \tag{2.7}$$

for any test functions φ, vanishing at the lateral boundary of the domain $\Omega_T\in\mathbb{R}^4$.

Here and throughout the text, M denotes the constants depending only on M_0 and T.

Next we put $\varphi(\boldsymbol{x},t)=\eta(t)\,\psi(\boldsymbol{x})$. Then

$$I=\int\int_{\Omega_T}c\,m(r)\eta\,\psi\,dxdt=\lim_{\varepsilon\to 0}\int\int_{\Omega_T}\widetilde{c}\,\chi^{\varepsilon}(\boldsymbol{x},t)\,\eta\,\psi\,dxdt. \tag{2.8}$$

Let

$$f^{\varepsilon}_{\psi}(t)=\int_{\Omega}\chi^{\varepsilon}(\boldsymbol{x},t)\widetilde{c}^{\varepsilon}\,\psi\,dx,\quad f_{\psi}(t)=\int_{\Omega}m(r)\,c\,\psi dx.$$

The limiting relation (2.8) means that

$$I=\lim_{\varepsilon\to 0}\int_0^T\eta(t)\,f^{\varepsilon}_{\psi}(t)dt=\int_0^T\eta(t)\,f_{\psi}(t)dt. \tag{2.9}$$

Coming back to the equality (2.7) with test functions of the form $\xi=\eta(t)\,\psi(\boldsymbol{x})$ we get

$$\int_0^T\big(\frac{d\eta}{dt}f^{\varepsilon}_{\psi}+\eta\,U^{\varepsilon}\big)dt=0,\ \ U^{\varepsilon}(t)=\int_{\Omega}\boldsymbol{u}^{\varepsilon}(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x})dx,$$
$$\int_0^T|U^{\varepsilon}(t)|^2dt\leqslant\|\boldsymbol{u}^{\varepsilon}\|^2_{2,\Omega_T}\|\nabla\psi\|^2_{2,Q}\leqslant M^2\,\|\nabla\psi\|^2_{2,Q}.$$

Thus,

$$\frac{df^{\varepsilon}_{\psi}}{dt}(t)=U^{\varepsilon}(t),\quad f^{\varepsilon}_{\psi}\in\mathbb{W}^1_2(0,T),$$
$$|f^{\varepsilon}_{\psi}(t)|\leqslant M_{\psi},\ \ |f^{\varepsilon}_{\psi}(t_1)-f^{\varepsilon}_{\psi}(t_2)|\leqslant M_{\psi}\,|t_1-t_2|^{\frac{1}{2}}$$

with some constant M_ψ, independent of ε.

The Arzela – Ascoli Theorem ([20] (Theorem 4, § 11, chapter III, p. 101) allows us to choose some subsequence $\{f_\psi^{\varepsilon_k}\}$, strongly convergent in the space $\mathbb{C}(\overline{\Omega})$ to some function $\overline{f}_\psi(t)$.

Passing to the limit as $\varepsilon_k \to 0$ in (2.9) $\{f_\psi^{\varepsilon_k}\}$ we arrive at the idewntity

$$\lim_{\varepsilon_k\to 0}\int_0^T \eta(t)\, f_\psi^{\varepsilon_k}(t)dt = \int_0^T \eta(t)\,\overline{f}_\psi(t)dt$$

for any test functions $\eta(t)$.

That is, $\overline{f}_\psi(t)$ coincides with $f_\psi(t)$ almost everywhere in $(0,T)$, which proves Lemma. □

Lemma 2.5. *In conditions of Theorem 2, there exists a subsequence $\{\varepsilon_k\}$ such that*

$$\lim_{\varepsilon_k\to 0}\varepsilon_k^2\int_\Omega |\nabla\,\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)|^2dx = 0 \tag{2.10}$$

for almost all $t_0 \in (0,T)$.

Proof. In fact, the uniform boundedness of the integrals

$$\int\int_{\Omega_T} |\nabla\widetilde{c}^{\varepsilon}(\boldsymbol{x},t)|^2dxdt$$

with respect to ε_k implies equality

$$\lim_{\varepsilon_k\to 0}\varepsilon_k^2\int\int_{\Omega_T} |\nabla\,\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t)|^2dxdt = 0. \tag{2.11}$$

Let

$$u_k(t_0) = \varepsilon_k^2\int_\Omega |\nabla\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)|^2dx.$$

Then (2.11) means, that the sequence $\{u_k\}$ converges strongly to zero in the space $\mathbb{L}_1(0,T)$.

Referring to Theorem 1 ([20], Chapter 37, § 1, p. 379), we conclude that there is a subsequence $\{u_k\}$, which converges to zero almost everywhere in $(0,T)$ as $k\to\infty$. □

Lemma 2.6. *In conditions of Theorem 2 any bounded in the space $\mathbb{L}_2(\Omega)$ sequence $\{\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\}$ converges weakly and two – scale in the space $\mathbb{L}_2(\Omega)$ $(0,T)$ to some function $c(\boldsymbol{x},t_0)$.*

Proof. Let the sequence ε_k be the same, as in Lemma 2.5 and $\Pi\subset(0,T)$ be the set of full measure for which the condition (2.10) holds true.

The boundedness of the sequence $\{\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\}$ in the space $\mathbb{L}_2(\Omega)$ implies the existence of some sequence (we leave for simplicity the same indices), which two – scale converges to 1 – periodic in the variable $\boldsymbol{y}$ function $C(\boldsymbol{x},t_0,\boldsymbol{y})$ from the space $\mathbb{L}_2(\Omega\times Y)$.

Integration by parts of the expression $\varepsilon_k\nabla\,\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\cdot\boldsymbol{\varphi}(\dfrac{\boldsymbol{x}}{\varepsilon_k})\psi(\boldsymbol{x})$ results the identity

$$\varepsilon_k\int_\Omega\nabla\,\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\cdot\boldsymbol{\varphi}(\frac{\boldsymbol{x}}{\varepsilon_k})\psi(\boldsymbol{x})dx=$$
$$-\,\varepsilon_k\int_\Omega\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\big(\boldsymbol{\varphi}(\frac{\boldsymbol{x}}{\varepsilon_k})\cdot\nabla\psi(\boldsymbol{x})\big)dx-\int_\Omega\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\big(\nabla\cdot\boldsymbol{\varphi}(\frac{\boldsymbol{x}}{\varepsilon_k})\big)\psi(\boldsymbol{x})dx,$$

which holds true for arbitrary functions $\boldsymbol{\varphi}\in\mathbb{W}^1_2(Y)$ and $\psi\in\overset{\circ}{\mathbb{W}}{}^1_2(\Omega)$.

Passing to the limit as $\varepsilon_k\to 0$ we get

$$\int_\Omega\psi(\boldsymbol{x})\big(\int_Y C(\boldsymbol{x},t_0,\boldsymbol{y})\nabla\cdot\boldsymbol{\varphi}(\boldsymbol{y})dy\big)dx=0.$$

Due to the arbitrary choice of functions $\boldsymbol{\varphi}$ and ψ, the last identity is equivalent to the equality $C(\boldsymbol{x},t_0,\boldsymbol{y})=c(\boldsymbol{x},t_0)$. □

Lemma 2.7. *In conditions of Theorem 2, the sequence* $\{\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t)\}$ *strongly converges in* $\mathbb{L}_2(\Omega_T)$ *to the function* $c(\boldsymbol{x},t)$.

Proof. To prove Lemma we put

$$\mathbb{H}^1=\overset{\circ}{\mathbb{W}}{}^1{}_2\,(\Omega)\subset\mathbb{H}^0=\mathbb{L}_2(\Omega)\subset\mathbb{H}^{-1}=\overset{\circ}{\mathbb{W}}{}^{-1}{}_2\,(\Omega),\;\; w_k(\boldsymbol{x},t_0)=\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)-c(\boldsymbol{x},t_0).$$

Taking into account inequality (9) ([29], § 12, Chapter II, p. 50) we obtain

$$\|w_k(.,t_0)\|^2_{\mathbb{H}^0}\leqslant\eta\,\|w_k(.,t_0)\|^2_{\mathbb{H}^1}+{}_\eta\,\|w_k(.,t_0)\|^2_{\mathbb{H}^{-1}}.$$

Next we integrate the last inequality with respect to time

$$\int_0^T\|w_k(.,t)\|^2_{\mathbb{H}^0}dt\leqslant\eta\int_0^T\|w_k(.,t)\|^2_{\mathbb{H}^1}dt+C_\eta\int_0^T\|w_k(.,t)\|^2_{\mathbb{H}^{-1}}dt.\quad(2.12)$$

and after that use a compact embedding of the space $\mathbb{H}^0(\Omega)$ into the space $\mathbb{H}^{-1}(\Omega)$ ([30], Theorem 3, § 2, Chapter 4): *the weak convergence of the sequence* $\{w_k(.,t)\}$ *in the space* $\mathbb{H}^0(\Omega)$ *implies the strong convergence this sequence in the space* $\mathbb{H}^{-1}(\Omega)$.

That is, $\lim\limits_{k\to\infty}\displaystyle\int_0^T\|w_k(.,t)\|^2_{\mathbb{H}^{-1}}dt=0.$

The last equality (2.12) and arbitrary choice of the positive number η means that $\lim\limits_{k\to\infty}\displaystyle\int_0^T\|w_k(.,t)\|^2_{\mathbb{H}^0}dt=0.$ □

2.5 Extension Lemmas.

The results on the extension of solutions of differential equations from one domain to another one are very important in the theory of homogenization of differential equations (see Zhikov et al. [10], Zhikov[11], Conca[31], Acerbi et al. [32]).

For example, if a sequence has different differential properties in different domains, and when homogenize it is desirable to preserve the best differential properties of solutions, then the better choice would be to extend the solution from the domain where it has the best differential properties onto the domain selected for homogenization.

All the results known up to now have been obtained for periodic structures. Fortunately,

the results of Lemma 1 ([10], §1 of Chapter III, p. 95) on "soft inclusions" are fully applicable to the case of the non-periodic structure with special periodicity of the pore space under consideration, when this structure is determined by the relations (2.1).

The structure that we consider here is, in principal, the "soft inclusion" and the special periodicity of the pore space does not play any role here because the extension is done for each individual cell $\Omega^{\boldsymbol{k},\varepsilon}\quad\Omega$ across the connected component of the boundary $\Gamma^{\boldsymbol{k},\varepsilon}(r)=\Gamma^{\varepsilon}(r)\cap\Omega^{\boldsymbol{k},\varepsilon}$ of the boundary $\Gamma^{\varepsilon}(r)$.

The following statement is true

Lemma 2.8. *Let $r\in\mathfrak{M}_T$ and functions $r(\boldsymbol{x},t)$ define the structure $\chi(r,\boldsymbol{y})$ of the domain $\Omega^{\varepsilon}_{f,T}$, given by relations* (2.1).

Then for any bounded in the space $\mathbb{W}^{1,0}_2(\Omega^{\varepsilon}_{f,T}(r))$ sequence $\{c^{\varepsilon}\}$ there exists an extension from the domain $\Omega^{\varepsilon}_{f,T}(r)$ onto domain Ω_T such that

$$\widetilde{c}^{\varepsilon}\in\mathbb{W}^{1,0}_2(\Omega_T),\ \ \|\widetilde{c}^{\varepsilon}\|_{2,\Omega_T}\leqslant M\,\|c^{\varepsilon}\|_{2,\Omega^{\varepsilon}_{f,T}(r)},$$
$$\|\nabla\widetilde{c}^{\varepsilon}\|_{2,\Omega_T}\leqslant M\,\|\nabla c^{\varepsilon}\|_{2,\Omega^{\varepsilon}_{f,T}(r)}. \tag{2.13}$$

Next, we will need functions that take the given values on the connected components $\Gamma^{\boldsymbol{k},\varepsilon}(r)$ of the boundary $\Gamma^{\varepsilon}(r)$. Namely, we will need an extension of the vector function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(\boldsymbol{x},t)$ from the boundary $\Gamma^{\varepsilon}(r)$, defined by function $r(\boldsymbol{x},t)$ by equalities (2.1), onto domain Ω_T.

At the same time, the connected components of the support of this func-

tion will be subsets of set $\Omega^{\boldsymbol{k},\varepsilon}$:

$$\begin{gathered}
\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)=\sum_{\boldsymbol{k}\in\mathbb{Z}}\overset{\circ}{\boldsymbol{v}}{}^{\boldsymbol{k},\varepsilon}(r;\boldsymbol{x},t),\ \ \overset{\circ}{\boldsymbol{v}}{}^{\boldsymbol{k},\varepsilon}\in\overset{\circ}{\mathbb{W}}{}_2^{1,0}(\Omega_T^{\boldsymbol{k},\varepsilon});\\
\|\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\|_{2,\Omega_T}+\varepsilon\,\|\mathbb{D}\big(x,\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\big)\|_{2,\Omega_T}\leqslant\varepsilon\,M\,M_0;\\
\nabla\cdot\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)=0,\ \ (\boldsymbol{x},t)\in\Omega^{\varepsilon}_{f,T},\\
\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)=0\ \ |\boldsymbol{\varsigma}(\frac{\boldsymbol{x}}{\varepsilon})|\geqslant\frac{5}{12};\\
\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x}_0,t)-\big(\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x}_0,t)\cdot\boldsymbol{N}^{\varepsilon}\big)\boldsymbol{N}^{\varepsilon}=0,\\
\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x}_0,t)\cdot\boldsymbol{N}^{\varepsilon}=-\boldsymbol{N}^{\varepsilon}\,D^{\varepsilon}_{N}((\boldsymbol{x}_0,t)),\ \ \boldsymbol{x}_0\in\Gamma^{\boldsymbol{k},\varepsilon}(r),
\end{gathered}\tag{2.14}$$

where $\boldsymbol{N}^{\varepsilon}$ is the unit normal to the boundary $\Gamma^{\varepsilon}(r)$ at the point $\boldsymbol{x}_0$ of this boundary at the time moment t, outward with respect to the domain $\Omega^{\varepsilon}_{f}(r)$ and D^{ε}_{N} is the velocity of the boundary $\Gamma^{\varepsilon}(r)$ in the direction of the vector $\boldsymbol{N}^{\varepsilon}$.

Lemma 2.9. *Let $r\in\mathfrak{M}_T$ and functions $r(\boldsymbol{x},t)$ define the structure $\chi(r,\boldsymbol{y})$ of the domain $\Omega^{\varepsilon}_{f,T}$, given by relations* (2.1)*.*

Then for all $\varepsilon>0$ there exist functions $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)$, satisfying the conditions (2.14)*.*

Proof. Let $r^0=\max\limits_{(\boldsymbol{x},t)\in\Omega_T}r(\boldsymbol{x},t)<\dfrac{1}{2}$ and infinitely smooth periodic function $\boldsymbol{\varsigma}_0(\boldsymbol{y})$ be selected from the conditions

$$\boldsymbol{\varsigma}_0(\boldsymbol{y})=\begin{cases}\boldsymbol{\varsigma}(\boldsymbol{y}) & |\boldsymbol{y}|<r^0,\\ 0 & |\boldsymbol{y}|>\frac{5}{12}\end{cases}$$

We consider a connected component $\Gamma^{\varepsilon,\boldsymbol{k}}(r)$ of the boundary $\Gamma^{\varepsilon}(r)$, where conditions (2.14):

$$|\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}_0}{\varepsilon})|=r(\boldsymbol{x}_0,t),\ \ -\frac{1}{\delta}\,\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x}_0,t)=D^{\varepsilon}_{N}(\boldsymbol{x}_0,t)\,\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}_0}{\varepsilon}),\ \ \boldsymbol{x}_0\in\Gamma^{\boldsymbol{k},\varepsilon}(r)\tag{2.15}$$

must be fulfilled.

We look for an extension $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)$ of the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x}_0,t)$ as

$$\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(r;\boldsymbol{x},t)=-\delta\,D^{\varepsilon}_{N}(\boldsymbol{x},t)\,\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\,u_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}),$$

where $\boldsymbol{x}_0=\varepsilon\,\boldsymbol{k}+\varepsilon\,\boldsymbol{\varsigma}(\dfrac{\boldsymbol{x}_0}{\varepsilon})$, $\boldsymbol{x}=\varepsilon\,\boldsymbol{k}+\varepsilon\,\boldsymbol{\varsigma}(\dfrac{\boldsymbol{x}}{\varepsilon})$ and the function to be find should be $u_t(\boldsymbol{x},\dfrac{\boldsymbol{x}}{\varepsilon})$.

We have to extend this function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\,(r;\boldsymbol{x}_0,t)$ from the boundary $\Gamma^{\varepsilon,\boldsymbol{k}}(r)$ into a "liquid component" of the domain $\Omega^{\boldsymbol{k},\varepsilon}$ so that the extended function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\,(r;\boldsymbol{x},t)$ should be solenoidal.

The required solenoidality of the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\,(r;\boldsymbol{x},t)$ provides the partial differential equation of the first order

$$
\begin{aligned}
0 = \nabla\cdot\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\,(r;\boldsymbol{x},t) = \\
u_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\Big(\big(\nabla D_N^{\varepsilon}(\boldsymbol{x},t)\cdot\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{\varepsilon}\,D_N^{\varepsilon}(\boldsymbol{x},t)\,\nabla_y\cdot\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\Big)+ \\
D_N^{\varepsilon}(\boldsymbol{x},t)\Big(\big(\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\cdot\nabla_x\,u_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{\varepsilon}\,\big(\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\cdot\nabla_y\,u_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\big)\Big)
\end{aligned}
\tag{2.16}
$$

for the function $u_t(\boldsymbol{x},\dfrac{\boldsymbol{x}}{\varepsilon})$, completed with initial condition

$$
u_t(\boldsymbol{x}_0,\frac{\boldsymbol{x}_0}{\varepsilon}) = 1 \tag{2.17}
$$

at the surface

$$
\Gamma^{\boldsymbol{k},\varepsilon}(r) = \{\boldsymbol{x}_0\in\Omega^{\boldsymbol{k},\varepsilon} : \boldsymbol{x}_0 = \varepsilon\,\boldsymbol{k}+\boldsymbol{\varsigma}(\frac{\boldsymbol{x}_0}{\varepsilon})\}. \tag{2.18}
$$

For a while, we will omit the upper index ε.

We rewrite equation (2.16) as

$$
\boldsymbol{a}_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\cdot\nabla_x\,u_t+\frac{1}{\varepsilon}\,\boldsymbol{a}_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\cdot\nabla_y\,u_t = b_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon})\,u_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}), \tag{2.19}
$$

where

$$
\begin{aligned}
b_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}) = -\Big(\big(\nabla_x\,D_N(\boldsymbol{x},t)\cdot\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{\varepsilon}\,D_N(\boldsymbol{x},t)\,\nabla_y\cdot\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon})\Big); \\
\boldsymbol{a}_t(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}) = D_N(\boldsymbol{x},t)\,\boldsymbol{\varsigma}_0(\frac{\boldsymbol{x}}{\varepsilon}),\ (\boldsymbol{x},t)\in\Omega_T.
\end{aligned}
\tag{2.20}
$$

Our immediate goal is to extend the function $u_t(\boldsymbol{x}_0,\dfrac{\boldsymbol{x}_0}{\varepsilon})$ from the surface $\Gamma^{\boldsymbol{k},\varepsilon}(r)$, given by the relations (2.18), into the liquid part $\Omega_f^{\boldsymbol{k},\varepsilon}(r)$ of the domain $\Omega^{\boldsymbol{k},\varepsilon}$ along characteristics of the partial differential equation of the first order (2.19).

Let s be the extension parameter, $\boldsymbol{x}_0$ be the starting point on the surface $\Gamma^{\boldsymbol{k},\varepsilon}(r)$ for the characteristic of equation (2.19).

According to [33], the Cauchy problem (2.17), (2.19) is equivalent to the Cauchy problem for the system of ordinary differential equations

$$
\begin{aligned}
\frac{\partial\boldsymbol{x}}{\partial s}(s,\boldsymbol{x}_0) = \boldsymbol{a}_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\boldsymbol{x}(s,\boldsymbol{x}_0)\big),\ \ \boldsymbol{x}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0; \\
\frac{\partial U_t}{\partial s}(s,\boldsymbol{x}_0) = b_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big), \\
U_t(0,\boldsymbol{x}_0) = 1 \text{ for } |\boldsymbol{\varsigma}(\frac{\boldsymbol{x}_0}{\varepsilon})| = r(\boldsymbol{x}_0,t)
\end{aligned}
\tag{2.21}
$$

for new unknown functions $\boldsymbol{x}(s,\boldsymbol{x}_0)$ and $U_t(s,\boldsymbol{x}_0)=u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\dfrac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)$.

Due to the uniform boundedness of the right hand – sides $\boldsymbol{a}_t(\boldsymbol{x},\dfrac{1}{\varepsilon}\,\boldsymbol{x})$ and $b_t(\boldsymbol{x},\dfrac{1}{\varepsilon}\,\boldsymbol{x})$ in the space $\mathbb{C}^1(\overline{Y}\times\overline{\Omega}_T)$, the Cauchy problem (2.21) has a unique solution $\{\boldsymbol{x}(s,\boldsymbol{x}_0),\,U_t(s,\boldsymbol{x}_0)\}$ for all $(\boldsymbol{x},t)\in\overline{\Omega}_T$ such that $U_t(s,\boldsymbol{x}_0)$ and $\dfrac{\partial U_t}{\partial s}(s,\boldsymbol{x}_0)$ are continuous functions on any bounded interval $s\in[0,S]$ [34] where

$$|\boldsymbol{\varsigma}_0\big(\frac{1}{\varepsilon}\,\boldsymbol{x}(S,\boldsymbol{x}_0)\big)|>0. \tag{2.22}$$

A natural question arises where the point $\boldsymbol{x}(s,\boldsymbol{x}_0)$ will start to move from the initial position with increasing parameter s? Into a liquid or a solid component? Since the vector $\boldsymbol{\varsigma}_0(\dfrac{\boldsymbol{x}}{\varepsilon})$ at the boundary point directed into the liquid component, then for $s>0$ the point $\boldsymbol{x}(s,\boldsymbol{x}_0)$ will begin to move into the liquid component with the growth of the parameter s: $\boldsymbol{x}(s,\boldsymbol{x}_0)\in\Omega_f^{\boldsymbol{k},\varepsilon}(t)$ for $s>0$.

Direct differentiation of equality $U_t(s,\boldsymbol{x}_0)=u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\dfrac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)$ with respect the parameter s gives us one more equality

$$\begin{aligned}\frac{\partial U_t}{\partial s}(s,\boldsymbol{x}_0)&=\nabla_x\,u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)\cdot\frac{\partial\boldsymbol{x}}{\partial s}(s,\boldsymbol{x}_0)+\\
&\frac{1}{\varepsilon}\,\nabla_y\,u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)\cdot\frac{\partial\boldsymbol{x}}{\partial s}(s,\boldsymbol{x}_0)=\\
&\nabla_x\,u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)\cdot\boldsymbol{a}_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\boldsymbol{x}(s,\boldsymbol{x}_0)\big)+\\
\frac{1}{\varepsilon}\,\nabla_y\,u_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),&\frac{1}{\varepsilon}\,\boldsymbol{x}(s,\boldsymbol{x}_0)\big)\cdot\frac{\partial\boldsymbol{x}}{\partial s}(s,\boldsymbol{x}_0)\cdot\boldsymbol{a}_t\big(\boldsymbol{x}(s,\boldsymbol{x}_0),\frac{1}{\varepsilon}\boldsymbol{x}(s,\boldsymbol{x}_0)\big).\end{aligned} \tag{2.23}$$

Thus, the second equation in (2.21) and equation (2.23) prove the validity of the relation (2.16)and, that is, the solenoidality of the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}\,(r;\boldsymbol{x},t)$ in the domain $\Omega_f^{\boldsymbol{k},\varepsilon}(r)$ everywhere wherever condition (2.22) is satisfied.

On the other hand-side, where this condition fails, one has that $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}$ $(r;\boldsymbol{x},t)=0$. So, indicated function will be solenoidal throughout the liquid component.

Since throughout the text of the article, the values of the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}$ $(r;\boldsymbol{x},t)$ are used only in the liquid component, the extension of this function into the solid component is standard. For example, along the normal to the boundary with further multiplication by a cutoff function. □

2.6 The Leray - Schauder fixed point theorem.

Definition 3. A continuous operator $\boldsymbol{T}$ acting from the Banach space $\mathbb{X}$ into itself (*notation*: $\boldsymbol{T}:\mathbb{X}\to\mathbb{X}$) is called ***Completely continuous*** if it maps every closed bounded set into a compact one.

Theorem 3. *Let the completely continuous operator $\boldsymbol{T}(\lambda,x)$, acting from the bounded Banach space $\mathbb{X}$ of elements $x\in\mathbb{X}$ into itself, continuously depends on the real parameter $\lambda\in[0,1]$ and let for $\lambda=0$ the operator $\boldsymbol{T}(0,x)$ there is a fixed point x_0: $\boldsymbol{T}(0,x_0)=x_0$.*

Then for all $\lambda\in[0,1]$, the operator $\boldsymbol{T}$ has at least one fixed point x_λ, such that $\boldsymbol{T}(\lambda,x_\lambda)=x_\lambda$.

3 The problem statement.

Let the function $r^\varepsilon(\boldsymbol{x},t)$ determine the structure of liquid and solid components in the domain Ω_T in according to the formulas (2.2), where $r_0(\boldsymbol{x})$ be a given function, $0\leqslant r^\varepsilon(\boldsymbol{x},t)\leqslant\dfrac{1}{2}$ and $\widetilde{r}^\varepsilon(\boldsymbol{x},t)=\max\{0,\,r_0(\boldsymbol{x})-r^\varepsilon(\boldsymbol{x},t)\}$, and

$$\boldsymbol{x}=\frac{\boldsymbol{x}'}{L},\;t=\frac{t'}{\tau},\;\boldsymbol{v}=\frac{\tau\,\boldsymbol{v}'}{L},\;p=\frac{p'}{p_a}\tag{3.1}$$

are dimensionless variables.

We look for the solution $\{\widetilde{r}^\varepsilon,\,\boldsymbol{v}^\varepsilon,\,p^\varepsilon,\,c^\varepsilon,\,c^\varepsilon_j\}$, $j=1,2,...,n$ to the system of differential equations

$$\nabla\cdot\big(\alpha^\varepsilon_\mu\,\mathbb{D}(x,\boldsymbol{v}^\varepsilon)-p^\varepsilon\,\mathbb{I}\big)=0,\;\;\nabla\cdot\boldsymbol{v}^\varepsilon=0,\tag{3.2}$$

$$\frac{\partial c^\varepsilon}{\partial t}=\nabla\cdot(d_0\nabla c^\varepsilon-\boldsymbol{v}^\varepsilon\,c^\varepsilon),\tag{3.3}$$

$$\frac{\partial c^\varepsilon_j}{\partial t}+\nabla\cdot(\boldsymbol{v}^\varepsilon c^\varepsilon_j)=0,\;j=1,...,k,\tag{3.4}$$

in unknown domain $\Omega^\varepsilon_{f,T}(\widetilde{r}^\varepsilon)$ describing the dynamics of the interaction of an acid solution in a carrier fluid with a concentration $c^\varepsilon(\boldsymbol{x},t)$ with a solid skeleton $\Omega^\varepsilon_{s,T}(\widetilde{r}^\varepsilon)$.

In (3.2) – (3.4) p_a – atmosphere pressure, $\boldsymbol{v}^\varepsilon$ and p^ε – velocity and pressure in the liquid, c^ε_j – concentrations of products of chemical reactions at the free boundary, $\Gamma^\varepsilon(\widetilde{r}^\varepsilon)=\partial\Omega^\varepsilon_f(\widetilde{r}^\varepsilon)\cap\partial\Omega^\varepsilon_s(\widetilde{r}^\varepsilon)$.

Generally speaking, the dimensionless diffusion coefficient d_0 can depend on the concentration of the acid. In our setting, we consider it to be a given positive constant.

At the free boundary $\Gamma^\varepsilon(\widetilde{r}^\varepsilon)$ hold true boundary condition (1.1) and conditions

$$v^\varepsilon_N=-\delta\,D^\varepsilon_N,\tag{3.5}$$

$$\boldsymbol{v}^\varepsilon = v^\varepsilon_N \boldsymbol{N}^\varepsilon, \tag{3.6}$$

$$(D^\varepsilon_N - v^\varepsilon_N)\, c^\varepsilon + d_0 \frac{\partial c^\varepsilon}{\partial N} = -\beta^\varepsilon\, c^\varepsilon, \tag{3.7}$$

$$(D^\varepsilon_N - v^\varepsilon_N)\, c^\varepsilon_j = -\beta^\varepsilon_j\, c^\varepsilon,\ j = 1, ..., n, \tag{3.8}$$

expressing the laws of conservation of mass and momentum [15] – [17].

Here $v^\varepsilon_N \overset{df}{=} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{N}^\varepsilon$ is a normal component of the liquid velocity $\boldsymbol{v}^\varepsilon$, $\dfrac{\partial c^\varepsilon}{\partial N} \overset{df}{=} \nabla c^\varepsilon \cdot \boldsymbol{N}^\varepsilon$, $\boldsymbol{N}^\varepsilon$ is a vector of the unit normal to the free boundary $\Gamma^\varepsilon(\widetilde{r}^\varepsilon)$, outward to the domain $\Omega^\varepsilon_f(\widetilde{r}^\varepsilon)$, $\delta = \dfrac{\rho_s - \rho_f}{\rho_f}$, ρ_s and ρ_f are densities of solid and liquid components respectively, β^ε and β^ε_j, $j = 1, ..., n$ are given positive constants.

Finally, at the given boundaries with injection wells S^1 and producing wells S^2, and at the impenetrable boundary S^0, the following conditions

$$\begin{aligned}\big(\alpha^\varepsilon_\mu\, \mathbb{D}\big(x, \boldsymbol{v}^\varepsilon(\boldsymbol{x}, t)\big) - p^\varepsilon(\boldsymbol{x}, t)\, \mathbb{I}\big) \cdot \boldsymbol{n}(\boldsymbol{x}) = \\ - p^j\, \boldsymbol{n}(\boldsymbol{x}),\ \boldsymbol{x} \in S^j,\ t > 0,\ j = 1, 2,\end{aligned} \tag{3.9}$$

$$\frac{\partial c^\varepsilon}{\partial n} = 0,\ \boldsymbol{x} \in S^0,\ t > 0, \tag{3.10}$$

$$\boldsymbol{v}^\varepsilon = 0,\ \ \boldsymbol{x} \in S^0,\ t > 0, \tag{3.11}$$

$$c^\varepsilon(\boldsymbol{x}, t) = c^0(\boldsymbol{x}),\ \ \boldsymbol{x} \in S^1 \cup S^2,\ t > 0, \tag{3.12}$$

$$c^\varepsilon_j(\boldsymbol{x}, t) = 0,\ j = 1, ..., n,\ \ \boldsymbol{x} \in S^1,\ t > 0 \tag{3.13}$$

are met.

In (3.7) – (3.13) $\boldsymbol{n}$ is the unit normal vector to the boundaries S^0, S^1 and S^2.

The problem (1.1), (3.1) – (3.13) is completed with initial conditions

$$c^\varepsilon(\boldsymbol{x}, 0) = c^0(\boldsymbol{x}),\ c^\varepsilon_j(\boldsymbol{x}, 0) = 0,\ j = 1, ..., n,\ \boldsymbol{x} \in \Omega, \tag{3.14}$$

$$\widetilde{r}^\varepsilon(\boldsymbol{x}, 0) = 0,\ \ \boldsymbol{x} \in \Omega. \tag{3.15}$$

Note, that the problem (1.1), (3.2), (3.3), (3.5) – (3.7), (3.9) – (3.12), (3.14), (3.15) on determining functions $\{\widetilde{r}^\varepsilon,\ \boldsymbol{v}^\varepsilon,\ p^\varepsilon,\ c^\varepsilon\}$ (let call it as **Problem** $\mathbb{A}^\varepsilon$) is solved independently of the problem on determining functions $c^\varepsilon_j(\boldsymbol{x}, t)$, $j = 1, 2, ..., n$.

The latter are determined after finding the functions $\{\boldsymbol{v}^\varepsilon,\ p^\varepsilon,\ c^\varepsilon\}$.

We will assume that $p^i = const$, $i = 1, 2$ and $p^0(x_1) = p^1(x_1 + \dfrac{1}{2}) + p^2(x_1 - \dfrac{1}{2})$.

For new unknown functions $\boldsymbol{v}^\varepsilon(\boldsymbol{x},t)$ and $\bar{p}^\varepsilon(\boldsymbol{x},t)=p^\varepsilon(\boldsymbol{x},t)-p^0(x_1)$ equations (3.2) in the domain $\Omega^\varepsilon_{f,T}$ will take the form

$$\nabla\cdot\big(\alpha^\varepsilon_\mu\,\mathbb{D}(x,\boldsymbol{v}^\varepsilon)-\bar{p}^\varepsilon\,\mathbb{I}\big)=\nabla\,p^0,\ \ (\boldsymbol{x},t)\in\Omega^\varepsilon_{f,T}(\widetilde{r}^{\,\varepsilon}),\tag{3.16}$$

$$\nabla\cdot\boldsymbol{v}^\varepsilon=0,\ \ (\boldsymbol{x},t)\in\Omega^\varepsilon_{f,T}(\widetilde{r}^{\,\varepsilon}),\tag{3.17}$$

and boundary condition (3.9) will be rewritten as

$$\big(\alpha^\varepsilon_\mu\,\mathbb{D}(x,\boldsymbol{v}^\varepsilon)-\bar{p}^\varepsilon\,\mathbb{I}\big)\cdot\boldsymbol{N}=0,\ \ \boldsymbol{x}\in S^j,\ j=1,2,\ t>0.\tag{3.18}$$

Let us transform the equation (3.3) and the boundary condition (3.7) so that the transformed equation and the boundary condition are equivalent to some integral identity.

Namely, we put

$$\frac{\partial c^\varepsilon}{\partial t}=\nabla\cdot\big(d_0\nabla c^\varepsilon-\boldsymbol{v}^\varepsilon\,(c^\varepsilon+\frac{\beta^\varepsilon}{\alpha^\varepsilon\,\delta})\big),\ \ (\boldsymbol{x},t)\in\Omega^\varepsilon_{f,T}(\widetilde{r}^{\,\varepsilon}),\tag{3.19}$$

$$d_0\,\frac{\partial c^\varepsilon}{\partial N}-v^\varepsilon_N\,(c^\varepsilon+\frac{\beta^\varepsilon}{\alpha^\varepsilon\,\delta})=0,\ \ (\boldsymbol{x},t)\in\Gamma^\varepsilon_T(\widetilde{r}^{\,\varepsilon}).\tag{3.20}$$

By its original definition, the acid concentration c^ε is always nonnegative, but its nonnegativity in our model does not follow from anywhere, since for the equation (3.19) and the boundary condition (3.20) there is no the maximum principle.

Therefore, we temporarily change the equation (3.19) to equation

$$\frac{\partial c^\varepsilon}{\partial t}=\nabla\cdot\big(d_0\nabla c^\varepsilon-\boldsymbol{v}^\varepsilon\,\psi(c^\varepsilon)\big),\ \ \boldsymbol{x}\in\Omega^\varepsilon_f(\widetilde{r}^{\,\varepsilon}),\ t>0,\tag{3.21}$$

and the boundary condition (3.20) to the boundary condition

$$d_0\frac{\partial c^\varepsilon}{\partial N}-v^\varepsilon_N\,\psi(c^\varepsilon)=-c^\varepsilon\,D^\varepsilon_N,\ \ \boldsymbol{x}\in\Gamma^\varepsilon(\widetilde{r}^{\,\varepsilon}),\ t>0.\tag{3.22}$$

In (3.22) $\boldsymbol{N}$ is a unit vector to the boundary $\Gamma^\varepsilon(\widetilde{r}^{\,\varepsilon})$ of outward to the domain $\Omega^\varepsilon_f(\widetilde{r}^{\,\varepsilon})$ and

$$\psi(s)=s+\frac{\beta^\varepsilon}{\alpha^\varepsilon\,\delta},\ \ 0<s<c^*\quad\psi(s)=0,\ \ s<0,\quad s>c^*.\tag{3.23}$$

We call the equation (3.21) as ***Modified diffusion – convection equation*** for the acid concentration, and the boundary condition (3.22) as ***Modified boundary condition*** for the acid concentration.

By problem ***Problem*** $\mathbb{B}^\varepsilon(r)$ we mean problem $\mathbb{A}^\varepsilon$ without boundary condition (1.1), but in a given domain $\Omega^\varepsilon_f(r)$ with a pore space structure $\chi(r,\boldsymbol{y})$, is determined by the function $r(\boldsymbol{x},t)$ in accordance with the relations (2.1).

Finally, by ***Dynamic problem*** $\mathbb{B}^\varepsilon(r)$ we mean initial – boundary value problem (3.11), (3.16) – (3.18), by ***Diffusion – convection problem*** $\mathbb{B}^\varepsilon(r)$ we mean initial – boundary value problem (3.10), (3.12), (3.14), (3.19), (3.20), and by ***Modified diffusion – convection problem*** $\mathbb{B}^\varepsilon(r)$ we mean initial – boundary value problem (3.10), (3.12), (3.14), (3.21), (3.22).

4 Equivalent integral identities. Formal homogenization.

To formulate main result we need to know homogenized models $\mathbb{H}$ and $\mathbb{H}(r)$. To get the differential equations of macroscopic mathematical models, we must somehow homogenize the microscopic mathematical models $\mathbb{A}^{\varepsilon}$ and $\mathbb{B}^{\varepsilon}(r)$.

Due to the remarks made earlier, it will be enough to homogenize only the model $\mathbb{B}^{\varepsilon}(r)$.

We assume that

$$\widetilde{r}^{\varepsilon} \in \mathfrak{M},\ \boldsymbol{v}^{\varepsilon},\ c^{\varepsilon} \in \mathbb{W}_2^{1,0}\big(\Omega^{\varepsilon}_{f,T}(r^{\varepsilon})\big)\ \ \bar{p}^{\varepsilon} \in \mathbb{L}_2(\Omega^{\varepsilon}_{f,T}),\ \ |p^0|^{(2)}_{\Omega} = M_0 < \infty.$$

Following the known schema [15] we extend solutions $\boldsymbol{v}^{\varepsilon}$ and c^{ε} onto the domain $\Omega^{\varepsilon}_{s,T}(r^{\varepsilon})$ and then write down the problem $\mathbb{A}^{\varepsilon}$ in the equivalent form of the corresponding integral identities so that the conditions on the free boundary will be included in the integral identities.

According to the lemma 2.9, the function c^{ε} is an extension of a function $\widetilde{c}^{\varepsilon}$, and the Lemma 2.8 was devoted to the extension of the function $\boldsymbol{v}^{\varepsilon}$.

By construction (see Lemma 2.9) the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}$ belongs to the space $\mathbb{W}_2^{1,0}(\Omega_T)$ and satisfies to the boundary conditions (3.4).

Now we put

$$\widetilde{\boldsymbol{v}}^{\varepsilon}(\boldsymbol{x},t) = \begin{cases} \boldsymbol{v}^{\varepsilon}(\boldsymbol{x},t) & \text{as } (\boldsymbol{x},t) \in \Omega^{\varepsilon}_{f,T}(\widetilde{r}^{\varepsilon}), \\ \overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}(\widetilde{r}^{\varepsilon};\boldsymbol{x},t) & \text{as } (\boldsymbol{x},t) \in \Omega^{\varepsilon}_{s,T}(\widetilde{r}^{\varepsilon}). \end{cases}$$

$$\widetilde{p}^{\varepsilon}(\boldsymbol{x},t) = \begin{cases} \bar{p}^{\varepsilon}(\boldsymbol{x},t) & \text{as } (\boldsymbol{x},t) \in \Omega^{\varepsilon}_{f,T}(\widetilde{r}^{\varepsilon}), \\ 0 & \text{as } (\boldsymbol{x},t) \in \Omega^{\varepsilon}_{s,T}(\widetilde{r}^{\varepsilon}). \end{cases} \tag{4.1}$$

Then the function $\widetilde{\boldsymbol{v}}^{\varepsilon} \in \mathbb{W}_2^{1,0}(\Omega_T)$ is an extension of the function $\boldsymbol{v}^{\varepsilon}$ from the domain $\Omega^{\varepsilon}_{f,T}(\widetilde{r}^{\varepsilon})$ onto the domain $\Omega^{\varepsilon}_{s,T}(\widetilde{r}^{\varepsilon})$ and satisfies boundary condition (4.1).

Definition 4. Let the structures $\chi^{\varepsilon}(\boldsymbol{x},t) = \chi\big(r(\boldsymbol{x},t), \dfrac{\boldsymbol{x}}{\varepsilon}\big)$ of the pore space $\Omega^{\varepsilon}_{f,T}(r)$ be defined by a function $r \in \mathfrak{M}_T$.

Then the solenoidal function $\widetilde{\boldsymbol{v}}^{\varepsilon} \in \mathbb{W}_2^{1,0}(\Omega_T)$ and functions $\widetilde{p}^{\varepsilon} \in \mathbb{L}_2(\Omega_T)$ and $\widetilde{c}^{\varepsilon} \in \mathbb{W}_2^{1,0}(\Omega_T)$ are called the ***Weak solution*** to the problem $\mathbb{A}^{\varepsilon}$, if the integral identities

$$\begin{aligned} &\int\!\!\int_{\Omega_T} \chi^{\varepsilon}\left(\alpha^{\varepsilon}_{\mu}\mathbb{D}(x,\widetilde{\boldsymbol{v}}^{\varepsilon}) - \widetilde{p}^{\varepsilon}\,\mathbb{I}\right) : \mathbb{D}(x,\boldsymbol{\varphi})\,dxdt = \\ &-\int\!\!\int_{\Omega_T} \nabla p^0 \cdot \boldsymbol{\varphi}\,dxdt - \int\!\!\int_{\Omega_T} (1-\chi^{\varepsilon})\sqrt{\alpha^{\varepsilon}_{\mu}}\ \overset{\circ}{\mathbb{F}}{}^{\varepsilon} : \mathbb{D}(x,\boldsymbol{\varphi})\big)\,dxdt, \end{aligned} \tag{4.2}$$

$$\int\int_{\Omega_T} \chi^\varepsilon \, (-\delta \, \frac{\partial \psi}{\partial t} + \widetilde{\boldsymbol{v}}^{\,\varepsilon} \cdot \nabla \psi \,) \, dxdt = 0, \tag{4.3}$$

$$\int_\Omega \chi^\varepsilon \, \widetilde{c}^\varepsilon(\boldsymbol{x}, t_0) \, \xi(\boldsymbol{x}, t_0) dx - \int_\Omega \chi^\varepsilon \, (\boldsymbol{x}, 0) \, \widetilde{c}^\varepsilon(\boldsymbol{x}, 0) \, \xi(\boldsymbol{x}, 0) dx +$$
$$\int_0^{t_0} \int_\Omega \chi^\varepsilon \Big(- \widetilde{c}^\varepsilon \frac{\partial \xi}{\partial \tau} + \nabla \xi \cdot \big(d_0 \, \nabla \, \widetilde{c}^\varepsilon - \widetilde{\boldsymbol{v}}^{\,\varepsilon} \, (\widetilde{c}^\varepsilon + \frac{\beta^\varepsilon}{\alpha^\varepsilon \, \delta})) \Big) dx d\tau = 0 \tag{4.4}$$

and boundary conditions (1.1) and (3.11) are fulfilled.

The identity (4.2) holds for all smooth functions $\boldsymbol{\varphi}$, vanishing at the boundary S^0 of the domain Ω for $t > 0$.

The identity (4.3) holds for all smooth functions ψ, vanishing at the boundary $S^1 \cup S^2$ of the domain Ω for $t > 0$ and at the initial moment of time.

The identity (4.4) holds for all smooth functions ξ, vanishing at the boundary $S^1 \cup S^2$ of the domain Ω for $t > 0$.

Remark 2. A little below we will show that the boundary condition (1.1) is equivalent to the is equivalent to the integral identity (4.25).

The identity (4.2) is equivalent to the dynamics Stokes equation (3.16).

In this identity

$$\overset{\circ}{\mathbb{F}}{}^{\varepsilon} = \sqrt{\alpha_\mu^\varepsilon} \, \mathbb{D}(x, \overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}) \;\; \overset{\circ}{\mathbb{F}}{}^{\varepsilon} \in \mathbb{L}_2(\Omega_T) \;\; \| \overset{\circ}{\mathbb{F}}{}^{\varepsilon} \|_{2, \Omega_T} \leqslant \; M \, M_0,$$

where the function $\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon}$ is defined by the formula (2.14).

The identity (4.3) is equivalent to the solenoidality of the function $\widetilde{\boldsymbol{v}}^{\,\varepsilon}(\boldsymbol{x}, t)$ and boundary conditions (3.5) and (3.6).

The identity (4.4) is equivalent to the diffusion – convection equation (3.21) and boundary and initial conditions (3.7), (3.10), (3.12), (3.14) and (3.22).

The equivalence of differential equations with corresponding boundary conditions to integral identities follows from the formula of integration by parts ([16], part II, §12) in the form

$$\int_0^{t_0} \int_{\Omega_f(t)} \xi \, (\frac{\partial A}{\partial t} + \nabla \cdot \boldsymbol{B}) dx dt + \int_0^{t_0} \int_{\Omega_f(t)} (A \frac{\partial \xi}{\partial t} + \boldsymbol{B} \cdot \nabla \xi) dx dt =$$
$$\int_{\Omega_f(t)} \xi(\boldsymbol{x}, t_0) \, A(\boldsymbol{x}, t_0) dx - \int_{\Omega_f(0)} \xi(\boldsymbol{x}, 0) \, A(\boldsymbol{x}, 0) dx +$$
$$\int_0^{t_0} \int_{\Gamma(t)} \xi(\boldsymbol{B} \cdot \boldsymbol{N} - A \, D_N) \, \sin \varphi \, d\sigma dt, \tag{4.5}$$

Here $\boldsymbol{N} \in \mathbb{R}^3$ is the unit normal vector to $\Gamma(t)$, pointing outward to $\Omega_f(t)$, D_N is the normal velocity of the boundary $\Gamma(t)$ in the direction of the normal

$\boldsymbol{N}$, and φ is the angle between unit vector $\boldsymbol{l}$ of the time axis and unit normal vector $\boldsymbol{\nu} \in \mathbb{R}^4$ to Γ_T, pointing outward to the domain $\Omega_{f,T}$, such that $\sin\varphi = \boldsymbol{\nu}\cdot\boldsymbol{N}$ and $\cos\varphi = \boldsymbol{\nu}\cdot\boldsymbol{l}$.

For example, in the identity (4.4) $A = \widetilde{c}^{\varepsilon}$ and $\boldsymbol{B} = \widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^{\varepsilon}+\dfrac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta}) - d_0\,\nabla\,\widetilde{c}^{\varepsilon}$.

The transformation of the diffusion – convection equation (3.3) leads to the equalities

$$
\begin{aligned}
0 = \int_0^{t_0}\int_{\Omega_f(t)} \xi\,\Big(\frac{\partial\widetilde{c}^{\varepsilon}}{\partial t} + \nabla\cdot\big(\widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta})\big) - d_0\,\nabla\,\widetilde{c}^{\varepsilon}\Big)dxdt = \\
\int_0^{t_0}\int_{\Omega_f(t)} \Big(\frac{\partial}{\partial t}(\widetilde{c}^{\varepsilon}\,\xi) - \nabla\cdot\big(\xi\,(d_0\,\nabla\,\widetilde{c}^{\varepsilon} - \widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta}))\big)\Big)dxdt + \\
\int_0^{t_0}\int_{\Omega_f(t)} \Big(-\frac{\partial\xi}{\partial t}\,c^{\varepsilon} + \nabla\xi\cdot\big(d_0\,\nabla\,\widetilde{c}^{\varepsilon} - \widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta})\big)\Big)dxdt = \\
\int_0^{t_0}\int_{\Gamma^{\varepsilon}(t)} \xi(-\widetilde{c}^{\varepsilon}\,D_N^{\varepsilon} + \widetilde{v}_N^{\varepsilon}\,(\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta}) - d_0\,\frac{\partial\widetilde{c}^{\varepsilon}}{\partial N})\,\sin\varphi\,d\sigma dt + \\
\int_0^{t_0}\int_{\Omega_f(t)} \Big(-\frac{\partial\xi}{\partial t}\,c^{\varepsilon} + \nabla\xi\cdot\big(d_0\,\nabla\,\widetilde{c}^{\varepsilon} - \widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}\,\delta})\big)\Big)dxdt.
\end{aligned}
$$

Integrand expression $\Psi_0 = -\widetilde{c}^{\varepsilon}\,D_N^{\varepsilon} + \widetilde{v}_N^{\varepsilon}\,(\widetilde{c}^{\varepsilon} + \dfrac{\theta}{\delta}) - d_0\,\dfrac{\partial\widetilde{c}^{\varepsilon}}{\partial N}$ in the integral over the free boundary is equal to zero due to the boundary conditions at the free boundary:

$$
D_N^{\varepsilon} = \alpha^{\varepsilon}\,\widetilde{c}^{\varepsilon},\;\; \widetilde{v}_N^{\varepsilon} = -\delta\,D_N^{\varepsilon} = -\delta\,\alpha^{\varepsilon}\,\widetilde{c}^{\varepsilon},\;\; -d_0\,\frac{\partial\widetilde{c}^{\varepsilon}}{\partial N} = \beta^{\varepsilon}\,\widetilde{c}^{\varepsilon} + D_N^{\varepsilon}\,\widetilde{c}^{\varepsilon} - \widetilde{v}_N^{\varepsilon}\,\widetilde{c}^{\varepsilon}.
$$

Definition 5. Let the structure $\chi(r,\boldsymbol{y})$ of the pore space $\Omega_{f,T}^{\varepsilon}(r)$ be given by the function $r \in \mathfrak{M}_T$.

Then functions $\widetilde{\boldsymbol{v}}^{\,\varepsilon}$ and $\widetilde{p}^{\varepsilon}$ are called the weak solution to the ***Dynamic problem*** $\mathbb{B}^{\varepsilon}(r)$ if integral identities (4.2) and (4.3) and the boundary condition (3.9)are fulfilled.

Definition 6. Let the structure $\chi(r,\boldsymbol{y})$ of the pore space $\Omega_{f,T}^{\varepsilon}(r)$ be given by the function $r \in \mathfrak{M}_T$ and function $\widetilde{\boldsymbol{v}}^{\,\varepsilon}$ be the weak solution to the dynamic problem $\mathbb{B}^{\varepsilon}(r)$.

Then the function $\widetilde{c}^{\varepsilon} \in \mathbb{V}_2(\Omega_T)$ is called the weak solution to the ***Diffusion – convection problem*** $\mathbb{B}^{\varepsilon}(r)$ if the integral identity (4.4) and boundary conditions (3.10) and (3.12) are fulfilled.

Definition 7. Let the structure $\chi(r,\boldsymbol{y})$ of the pore space $\Omega_{f,T}^{\varepsilon}(r)$ be given by the function $r \in \mathfrak{M}_T$ and function $\widetilde{\boldsymbol{v}}^{\,\varepsilon}$ be the weak solution to the dynamic problem $\mathbb{B}^{\varepsilon}(r)$.

Then the function $\widetilde{c}^\varepsilon \in \mathbb{V}_2(\Omega_T)$ is called the weak solution to the ***Modified diffusion – convection problem*** $\mathbb{B}^\varepsilon(r)$ if the integral identity

$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\Big(-\widetilde{c}^\varepsilon\frac{\partial\xi}{\partial t}+\nabla\xi\cdot\big(d_0\,\nabla\,\widetilde{c}^\varepsilon-\widetilde{\boldsymbol{v}}^\varepsilon\psi(\widetilde{c}^\varepsilon)\big)\Big)dxdt+$$
$$\int_\Omega\chi^\varepsilon(\boldsymbol{x},t_0)\,\widetilde{c}^\varepsilon(\boldsymbol{x},t_0)\,\xi(\boldsymbol{x},t_0)dx-\int_\Omega\chi^\varepsilon\,(\boldsymbol{x},0)\,\widetilde{c}^\varepsilon(\boldsymbol{x},0)\,\xi(\boldsymbol{x},0)dx=0 \quad (4.6)$$

and boundary conditions (3.10)and (3.12) are fulfilled for all t_0, $0<t_0<T$ and for all smooth functions ξ, vanishing at the boundary S^0.

(4.6) $\psi(s)$ (3.23).

Definition 8. Let the structure $\chi(r,\boldsymbol{y})$ of the pore space $\Omega^\varepsilon_{f,T}(r)$ be given by the function $r\in\mathfrak{M}_T$ and function $\widetilde{\boldsymbol{v}}^\varepsilon$ be the weak solution to the dynamic problem $\mathbb{B}^\varepsilon(r)$,and the function $\widetilde{c}^\varepsilon$ be the weak solution to the diffusion – convection problem $\mathbb{B}^\varepsilon(r)$.

Then the triple of functions $\{\widetilde{\boldsymbol{v}}^\varepsilon,\,\widetilde{p}^\varepsilon,\,\widetilde{c}^\varepsilon\}$ is called the ***Weak solution to the problem*** $\mathbb{B}^\varepsilon(r)$.

Lemma 4.1. *Let the structure $\chi(r,\boldsymbol{y})$ of the pore space $\Omega^\varepsilon_{f,T}(r)$ be given by the function $r\in\mathfrak{M}_T$ and the solution $\{\widetilde{\boldsymbol{v}}^\varepsilon,\,\widetilde{p}^\varepsilon,\,\widetilde{c}^\varepsilon\}$ to the problem $\mathbb{B}^\varepsilon(r)$ exists and has the smoothness necessary for the selection of convergent subsequences:*

$$\widetilde{p}^\varepsilon,\ \widetilde{\boldsymbol{v}}^\varepsilon,\ \varepsilon\,\mathbb{D}(x,\widetilde{\boldsymbol{v}}^\varepsilon)\in\mathbb{L}_2(\Omega_T),\ \widetilde{c}^\varepsilon\in\mathbb{W}^{1,0}_2(\Omega_T),$$
$$\frac{\partial\widetilde{c}^\varepsilon}{\partial t}\in\mathbb{L}_2\big(0,T;\mathbb{W}^{-1}_2(\Omega)\big);$$
$$\|\widetilde{p}^\varepsilon\|_{2,\Omega_T}+\|\widetilde{\boldsymbol{v}}^\varepsilon\|_{2,\Omega_T}+\|\sqrt{\alpha_\mu}\,\mathbb{D}(x,\widetilde{\boldsymbol{v}}^\varepsilon)\|_{2,\Omega_T}+$$
$$\|\nabla\,\widetilde{c}^\varepsilon\|_{2,\Omega_T}+\|\frac{\partial\widetilde{c}^\varepsilon}{\partial t}\|_{W^{-1}}\leqslant M\,M_0. \quad (4.7)$$

As M here here and everywhere else, we will denote constants that do not depend on ε.

Then there exist some subsequences of the sequences $\{\widetilde{\boldsymbol{v}}^\varepsilon\}$, $\{\widetilde{p}^\varepsilon\}$ and $\{\widetilde{c}^\varepsilon\}$ such that

$$\widetilde{\boldsymbol{v}}^\varepsilon(\boldsymbol{x},t)\rightharpoonup\boldsymbol{v}(\boldsymbol{x},t),\ \ \widetilde{\boldsymbol{v}}^\varepsilon(\boldsymbol{x},t)\stackrel{t.-s.}{\to}\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y}),$$
$$\varepsilon\mathbb{D}\big(x,\widetilde{\boldsymbol{v}}^\varepsilon(\boldsymbol{x},t)\big)\stackrel{t.-s.}{\to}\mathbb{D}\big(y,\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})\big);$$
$$\widetilde{p}^\varepsilon(\boldsymbol{x},t)\rightharpoonup p(\boldsymbol{x},t),\ \ \widetilde{p}^\varepsilon(\boldsymbol{x},t)\stackrel{t.-s.}{\to}P_f\,(\boldsymbol{x},t,\boldsymbol{y})\,\chi(\boldsymbol{x},t,\boldsymbol{y});$$
$$\overset{\circ}{\boldsymbol{v}}{}^\varepsilon\,(\boldsymbol{x},t)\ \textit{strongly converges in}\ \mathbb{L}_2(\Omega_T)\ \textit{to zero};$$
$$\overset{\circ}{\mathbb{F}}{}^\varepsilon\,(\boldsymbol{x},t)\ \textit{strongly converges in}\ \mathbb{L}_2(\Omega_T)\ \textit{to zero};$$
$$\widetilde{c}^\varepsilon(\boldsymbol{x},t)\ \textit{strongly converges in}\ \mathbb{L}_2(\Omega_T)\ \textit{to the function}\ c(\boldsymbol{x},t);$$
$$\mathbb{D}(x,\widetilde{c}^\varepsilon)\stackrel{t.-s.}{\to}\mathbb{D}(x,c)+\mathbb{D}\big(y,C(\boldsymbol{x},t,\boldsymbol{y})\big)$$

as $\varepsilon \to 0$.

Here

$$\boldsymbol{v} \in \mathbb{L}_2(\Omega_T),\ c,\, p \in \mathbb{W}^{1,0}_2(\Omega_T),\ \ \Pi_f,\, \boldsymbol{V},\, C,\, \mathbb{D}(y,\boldsymbol{V}),\, \mathbb{D}(y,C) \in \mathbb{L}_2(\Omega_T \times Y).$$

The proof of the Lemma follows from the results of § 2 on the choice of convergent subsequences. For the limiting procedure, it is necessary to know the behavior of the coefficients $\alpha^{\varepsilon}_{\mu}$,

The limit as $\varepsilon \to 0$ in (4.3) is quite standard (see [15], pp. 6 – 11):

$$\boldsymbol{v} = -\frac{1}{\mu_1}\mathbb{C}^v(r)\cdot\nabla\,(p+p^0),\ \ (\boldsymbol{x},t)\in\Omega_T. \tag{4.8}$$

$$\nabla\cdot\boldsymbol{v} = \delta\,\frac{\partial\, m}{\partial t},\ \ (\boldsymbol{x},t)\in\Omega_T. \tag{4.9}$$

At the same time identities (4.2) and (4.3) contain boundary conditions

$$v_n = \boldsymbol{v}\cdot\boldsymbol{n} = 0,\ \ \boldsymbol{x}\in S^0,\ \ p=0,\ \boldsymbol{x}\in S^1\cup S^2,\ \ t>0. \tag{4.10}$$

on the given boundaries.

The limit in the identity (4.4) as $\varepsilon \to 0$ gives us the identity

$$\int_\Omega m(\boldsymbol{x},t_0)\,c(\boldsymbol{x},t_0)\,\xi(\boldsymbol{x},t_0)dx - \int_\Omega m(\boldsymbol{x},0)\,c^0(\boldsymbol{x})\,\xi(\boldsymbol{x},0)dx +$$
$$\int_0^{t_0}\int_\Omega\Big(-\,m\,c\frac{\partial\xi}{\partial t} + \nabla\xi\cdot\big(d_0\,\mathbb{C}^c(r)\cdot\nabla\,c - \boldsymbol{v}\,(c+\frac{\theta}{\delta})\big)\Big)dxd\tau = 0, \tag{4.11}$$

which holds true for all smooth functions ξ, vanishing on the boundary $S^1\cup S^2$ of the domain Ω for $t\geqslant 0$.

The identity (4.17) is formally equivalent to the ***Homogenized diffusion – convection equation***

$$\frac{\partial}{\partial t}(m\,c) = \nabla\cdot\big(d_0\,\mathbb{C}^c(r)\cdot\nabla\,c - \boldsymbol{v}\,(c+\frac{\theta}{\delta})\big). \tag{4.12}$$

β^{ε} and α^{ε} as $\varepsilon \to 0$.

The constants β^{ε} will be defined in (4.23).

In the present publication we restrict ourself with the case

$$\mu_0 = 0\ \ 0<\mu_1<\infty,\ \ \lim_{\varepsilon\to 0}\frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}} = \theta = \mathrm{const} \tag{4.13}$$

of ***Weakly viscous liquid*** (see [15]).

The limit as $\varepsilon \to 0$ in (4.3) is quite standard (see [15], pp. 6 – 11):

$$\boldsymbol{v} = -\frac{1}{\mu_1}\mathbb{C}^v(r)\cdot\nabla\,(p+p^0),\ \ (\boldsymbol{x},t)\in\Omega_T. \tag{4.14}$$

$$\nabla\cdot\boldsymbol{v}=\delta\,\frac{\partial\,m}{\partial t},\ \ (\boldsymbol{x},t)\in\Omega_T. \tag{4.15}$$

At the same time identities (4.2) and (4.3) contain boundary conditions

$$v_n=\boldsymbol{v}\cdot\boldsymbol{n}=0,\ \ \boldsymbol{x}\in S^0,\ \ p=0,\ \ \boldsymbol{x}\in S^1\cup S^2,\ \ t>0. \tag{4.16}$$

on the given boundaries.

The limit in the identity (4.4) as $\varepsilon\to 0$ gives us the identity

$$\int_\Omega m(\boldsymbol{x},t_0)\,c(\boldsymbol{x},t_0)\,\xi(\boldsymbol{x},t_0)dx-\int_\Omega m(\boldsymbol{x},0)\,c^0(\boldsymbol{x})\,\xi(\boldsymbol{x},0)dx+$$
$$\int_0^{t_0}\int_\Omega\Big(-m\,c\frac{\partial\xi}{\partial t}+\nabla\xi\cdot\big(d_0\,\mathbb{C}^c(r)\cdot\nabla c-\boldsymbol{v}\,(c+\frac{\theta}{\delta})\big)\Big)dxd\tau=0, \tag{4.17}$$

which holds true for all smooth functions ξ, vanishing on the boundary $S^1\cup S^2$ of the domain Ω for $t\geqslant 0$.

The identity (4.17) is formally equivalent to the ***Homogenized diffusion – convection equation***

$$\frac{\partial}{\partial t}(m\,c)=\nabla\cdot\big(d_0\,\mathbb{C}^c(r)\cdot\nabla\,c-\boldsymbol{v}\,(c+\frac{\theta}{\delta})\big). \tag{4.18}$$

Symmetric strictly positive definite matrices $\mathbb{C}^v(r)$ and $\mathbb{C}^c(r)$ depend on the structure of the pore space $Y_f(r)$ and are defined by formulas (1.1.27) and (10.1.61) in [15] accordingly:

$$\mathbb{C}^v(r)=2\int_{Y_f(r)}\sum_{i=1}^3\boldsymbol{V}^i(r;\boldsymbol{y})\otimes\boldsymbol{e}^i\big)dy,$$
$$\triangle_y\boldsymbol{V}^i-\nabla_y\Pi^i=-\boldsymbol{e}^i,\ \ \nabla_y\cdot\boldsymbol{V}^i=0,\ \ |\boldsymbol{y}|>r,$$
$$\boldsymbol{V}^i(r;\boldsymbol{y})=0,\quad |\boldsymbol{y}|=r, \tag{4.19}$$

$$\mathbb{C}^c(r)=m\,\mathbb{I}+\mathbb{C}^c_0(r),\ \ \mathbb{C}^c_0(r)=\int_{Y_f(r)}\big(\sum_{i=1}^3\nabla_yC^i(r;\boldsymbol{y})\otimes\boldsymbol{e}^i\big)dy,$$
$$\triangle_yC^i=0,\ \ |\boldsymbol{y}|>r,\ \ \int_{Y_f(r)}C^i(r;\boldsymbol{y})dy=0,$$
$$(\nabla_yC^i+\boldsymbol{e}^i)\cdot\boldsymbol{n}=0,\ \ |\boldsymbol{y}|=r, \tag{4.20}$$

where $\{\boldsymbol{e}^1,\,\boldsymbol{e}^2,\,\boldsymbol{e}^3\}$ is standard orthonormal basis in $\mathbb{R}^3$.

Differential equations (4.14), (4.15) and (4.18) are completed with boundary conditions (4.16) and boundary and initial conditions

$$\frac{\partial c}{\partial n}=\nabla c\cdot\boldsymbol{n}=0,\ \ \boldsymbol{x}\in S^0,\ \ c=c^0,\ \ \boldsymbol{x}\in S^1\cup S^2,\ \ t>0, \tag{4.21}$$

$$c(\boldsymbol{x},0)=c^0,\ \ \boldsymbol{x}\in\Omega. \tag{4.22}$$

We call as the ***Problem*** $\mathbb{H}(r)$ the initial boundary – value problem (4.14) – (4.22) for the given structure $\chi(\boldsymbol{x},t,\boldsymbol{y})$ of the pore space, defined by the function $r\in\mathfrak{M}_T$.

The boundary – value problem (4.14) – (4.16) we call as the ***Dynamic problem*** $\mathbb{H}(r)$.

Finally, the initial boundary – value (4.18) (4.20) – (4.22) we call as ***Diffusion – convection problem*** $\mathbb{H}(r)$.

Everywhere below, where it does not cause discrepancies, we will write $\mathbb{C}^v$ and $\mathbb{C}^c$ instead of $\mathbb{C}^v(r)$ and $\mathbb{C}^c(r)$.

When deriving the problem $\mathbb{H}(r)$ we did not use anywhere an additional boundary condition (1.1) on the free boundary $\Gamma(r)$.

That is, the problem $\mathbb{H}(r)$ is a homogenization of the problem $\mathbb{B}^\varepsilon(r)$.

Thus, to derive the **Problem** $\mathbb{H}$ as a homogenization of the problem $\mathbb{A}^\varepsilon$ we simply complete the problem $\mathbb{H}(r)$ with condition, which is a homogenization of the additional boundary condition (1.1) on the free boundary. After that we find the structure χ^ε of the pore space according to the formulas(2.1).

To homogenize boundary condition (1.1) we will need the following

Assumption 1. For the problem $\mathbb{A}^\varepsilon$ hold true conditions

$$\alpha^\varepsilon=\varepsilon\,\theta,\ \ \beta^\varepsilon=\varepsilon, \tag{4.23}$$

where θ is a given positive constant.

Lemma 4.2. *Let functions $\widetilde{c}^\varepsilon(\boldsymbol{x},t)$ be extensions of functions $c^\varepsilon(\boldsymbol{x},t)$ (see. Lemma 2.8), the sequence $\{\widetilde{c}^\varepsilon\}$ be bounded in the space $\mathbb{W}^{1,0}_2(\Omega_T)$ and $\lim\limits_{\varepsilon\to 0}\|\widetilde{c}^\varepsilon - c\|_{2,\Omega_T}=0$.*

Then under condition of the Assumption 1 the equality

$$d_n(\boldsymbol{x},t)=\theta\,c(\boldsymbol{x},t),\ \ \boldsymbol{y}\in\gamma(\boldsymbol{x},t)\subset Y. \tag{4.24}$$

will be a homogenization of the additional boundary condition (1.1)

Proof. The additional boundary condition (**??**) on the boundary $\Gamma^\varepsilon_c(r)$ is equivalent to the integral identity

$$\int\int_{\Omega_{t_0}}\chi^\varepsilon\big(-\frac{\partial}{\partial t}(\zeta\,\boldsymbol{a}^\varepsilon\cdot\boldsymbol{\xi}^\varepsilon_0)+\varepsilon\,\nabla\cdot(\zeta\,\widetilde{c}^\varepsilon\,\boldsymbol{\xi}^\varepsilon_0)\big)dxdt=0 \tag{4.25}$$

which is valid for any smooth functions $\boldsymbol{\xi}^\varepsilon_c(r,\boldsymbol{x})=\boldsymbol{\xi}_c(r,\dfrac{\boldsymbol{x}}{\delta})$, functions ζ, vanishing at $t=0$ and at $t=t_0$ and at boundary $\partial\Omega$, and functions $\boldsymbol{a}^\varepsilon_c(r,\boldsymbol{x})=\boldsymbol{a}_c(r,\dfrac{\boldsymbol{x}}{\delta})$, such that $\boldsymbol{a}_c$ vanishes outside of some small neighbourhood of $\gamma_c(r)$ and $\boldsymbol{a}_c(r,\boldsymbol{y})=\boldsymbol{n}_c(r)$, where $\boldsymbol{n}_c(r)$ is the unit normal to $\gamma_c(r)$, outward to the domain $Y_f(r)$.

In fact, let $\varsigma(\boldsymbol{x},t)$ be an arbitrary function, vanishing at $t=0$ and at $t=t_0$. Then, according to the identity (4.5) of integration by parts with $A=1$ and $\boldsymbol{B}=0$, we have

$$\int_0^{t_0}\int_{\Omega_f^{\varepsilon}(r)}\frac{\partial u}{\partial t}dxdt=-\int\int_{\Gamma^{\varepsilon}(r)}D_N^{\varepsilon}\,u\,\sin\varphi\,d\sigma\,dt.$$

In the last identityφ is the angle between the time axis and the unit normal $\boldsymbol{N}^{\varepsilon}$ to the boundary $\Gamma^{\varepsilon}(r)$ outward with respect to $\Omega_f^{\varepsilon}(r)$, D_N^{ε} is the velocity of $\Gamma^{\varepsilon}(r)$ in the direction $\boldsymbol{N}^{\varepsilon}$.

Next in the integral over the surface $\Gamma_T^{\varepsilon}(r)$ we use boundary condition (1.1) and Assumption 1:

$$\int\int_{\Omega_{t_0}}\chi^{\varepsilon}\frac{\partial u}{\partial t}dxdt=-\int\int_{\Gamma_{t_0}^{\varepsilon}(r)}\alpha^{\varepsilon}\,\widetilde{c}^{\varepsilon}\,u\,\sin\varphi\,d\sigma\,dt=-\int\int_{\Gamma_{t_0}^{\varepsilon}(r)}\varepsilon\,\theta\,\widetilde{c}^{\varepsilon}\,u\,\sin\varphi\,d\sigma dt.$$

Let now

$$u=\boldsymbol{a}^{\varepsilon}(\boldsymbol{x},t)\cdot\boldsymbol{\xi}^{\varepsilon}(\boldsymbol{x},t),\;\;\boldsymbol{a}^{\varepsilon}(\boldsymbol{x},t)=\boldsymbol{a}(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}),\;\;\boldsymbol{\xi}^{\varepsilon}(\boldsymbol{x},t)=\zeta(\boldsymbol{x})\,\boldsymbol{\xi}_0(\frac{\boldsymbol{x}}{\varepsilon},t),$$

with 1 – periodic in $\boldsymbol{y}$ functions $\boldsymbol{a}(\boldsymbol{x},t,\boldsymbol{y})$ and $\boldsymbol{\xi}_0(\boldsymbol{y},t)$ such that $\boldsymbol{a}(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{n}(\boldsymbol{x},t,\boldsymbol{y})$ for $\boldsymbol{y}\in\gamma(\boldsymbol{x},t)$.

One has

$$\int\int_{\Omega_{t_0}}\chi^{\varepsilon}\frac{\partial}{\partial t}(\boldsymbol{a}^{\varepsilon}\cdot\boldsymbol{\xi}^{\varepsilon})dxdt+\int_0^{t_0}\int_{\Gamma^{\varepsilon}(r)}\varepsilon\,\theta\,\widetilde{c}^{\varepsilon}(\boldsymbol{n}\cdot\boldsymbol{\xi}^{\varepsilon})\,\sin\varphi\,d\sigma dt=$$
$$\int\int_{\Omega_{t_0}}\chi^{\varepsilon}\big(\frac{\partial}{\partial t}(\zeta\,\boldsymbol{a}^{\varepsilon}\cdot\boldsymbol{\xi}_0^{\varepsilon})+\varepsilon\,\theta\,\nabla\cdot(\zeta\,\widetilde{c}^{\varepsilon}\,\boldsymbol{\xi}_0^{\varepsilon})\big)dxdt=0.$$

The obtained identity

$$\int\int_{\Omega_{t_0}}\chi^{\varepsilon}\big(\frac{\partial}{\partial t}(\zeta\,\boldsymbol{a}^{\varepsilon}\cdot\boldsymbol{\xi}_0^{\varepsilon})+\varepsilon\,\theta\,\nabla\cdot(\zeta\,\widetilde{c}^{\varepsilon}\,\boldsymbol{\xi}_0^{\varepsilon})\big)dxdt=0 \tag{4.26}$$

is equivalent to the boundary condition (1.1).

It is valid for any functions $\boldsymbol{a}^{\varepsilon}(\boldsymbol{x},t)=\boldsymbol{a}(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})$, $\boldsymbol{a}(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{n}(\boldsymbol{x},t,\boldsymbol{y})$, where $\boldsymbol{n}(\boldsymbol{x},t,\boldsymbol{y})$is a unit normal vector to $\gamma(\boldsymbol{x},t)$, outward with respect to the domain $Y_{r,f}(\boldsymbol{x},t)$, arbitrary functions $\boldsymbol{\xi}_0(\boldsymbol{y},t)$, equal to zero at $t=0$ and at $t=t_0$, and arbitrary functions $\zeta(\boldsymbol{x})$, vanishing on the boundary $\partial\Omega$.

Passing to the limit as $\varepsilon\to 0$ in (4.26) we get

$$\int_{\Omega}dx\,\zeta\int_0^{t_0}\int_{Y_f(r)}\Big(\frac{\partial}{\partial t}\big(\boldsymbol{a}\cdot\boldsymbol{\xi}_0\big)+\nabla_y\cdot(\theta\,c\,\boldsymbol{\xi}_0)\Big)dydt=0.$$

Here we have used the last statement of the Theorem 1 on a two – scale convergence of functions from $\mathbb{W}_2^{1,0}(\Omega_T)$ to the function independent of the fast variable $\boldsymbol{y}$.

The equality (4.24) we need follows from the last identity for the arbitrary functions ζ and from the identity (4.5):

$$0=\int_0^{t_0}\int_{Y_f(r)}\Big(\frac{\partial}{\partial t}\big(\boldsymbol{a}\cdot\boldsymbol{\xi}_0\big)+\nabla_y\cdot(c\,\boldsymbol{\xi}_0)\Big)dydt=$$
$$\int_0^{t_0}\int_\gamma(-d_n+\theta\,c)(\boldsymbol{n}\cdot\boldsymbol{\xi}_0)\,\sin\varphi\,d\sigma dt=0.$$

$\square$

Turning to the boundary condition (4.24) we see that its fulfillment entails equality

$$\frac{\partial r}{\partial t}(\boldsymbol{x},t)=-\theta\,c(\boldsymbol{x},t).$$

The last relation suggests a scheme for proving the existence of a solution to the problem $\mathbb{H}$ using Schauder's fixed point theorem.

We fix the set $\mathfrak{M}_T$ of functions $r(\boldsymbol{x},t)$, defining the pore space $\Omega_f^\varepsilon(r)$ in the variables $(\boldsymbol{x},t)$ and the pore space $Y_f(r)$ in the variables $(\boldsymbol{y},t)$.

For given structure of the pore space we find the solution $\{\boldsymbol{v}=\mathbb{F}^v(r),\,\nabla p=\mathbb{F}^p(r),\,c=\mathbb{F}^c(r)\}$ to the problem $\mathbb{H}(r)$ and function

$$R(\boldsymbol{x},t_0)=r_0(\boldsymbol{x})-\theta\int_0^{t_0}c(\boldsymbol{x},t)dt\,\equiv\mathbb{F}(r),\tag{4.27}$$

by which a new structure of the pore space is constructed in accordance with the formulas (2.2).

The fixed points $r_*(\boldsymbol{x},t)$ of the operator $\mathbb{F}(r)$ define the characteristic function $\chi(r_*,\boldsymbol{y})$ of the pore space $Y_f(r_*)$ according to the formula (2.2) § 2, for which the problem $\mathbb{H}(r_*)$ coincides with the problem $\mathbb{H}$.

Lemma 4.3. *Let*

$$\nabla m,\,\frac{\partial m}{\partial t}\in\mathbb{L}_\infty(\Omega_{t_0}),$$
$$\mu_*|\boldsymbol{z}|^2\leqslant\big(\mathbb{C}^v\cdot\boldsymbol{z}\big)\cdot\boldsymbol{z}\leqslant\mu^*|\boldsymbol{z}|^2,\;\;\mu_*|\boldsymbol{z}|^2\leqslant\big(\mathbb{C}^c\cdot\boldsymbol{z}\big)\cdot\boldsymbol{z}\leqslant\mu^*|\boldsymbol{z}|^2$$

and $p,\,c\in\mathbb{L}_\infty(\Omega_T)\cap\mathbb{W}_2^{1,0}(\Omega_T)$, $\boldsymbol{v}\in\mathbb{L}_\infty(\Omega_T)$ *are the weak solution to the problem* $\mathbb{H}(r)$.

Then operators $\mathbb{F}^c(r)$, $\mathbb{F}^v(r)$ *and* $\mathbb{F}^p(r)$ *are defined correctly.*

Proof. The difference $\widetilde{p} = p_1 - p_2$ of two possible weak solutions p_1 and p_2 of the problem (4.14) – (4.16) satisfies in the domain Ω_{t_0} the homogeneous elliptic equation

$$\nabla \cdot \big(\mathbb{C}^v \cdot \nabla \widetilde{p}\big) = 0 \tag{4.28}$$

and homogeneous boundary conditions (4.16)at the boundary S of the domain Ω for all t, $0 < t < t_0$.

Strict positive definiteness of the matrix $\mathbb{C}^v$, guarantees the uniqueness of the solution of the problem (4.16), (4.28) and correct definition of operators $\mathbb{F}^v(r)$ and $\mathbb{F}^p(r)$.

Now consider the difference $u = m_1\, c_1 - m_2\, c_2$, $m_i = m(r_i)$, $i = 1, 2$, two possible solutions c_1 and c_2 to the problem (4.18), (4.21), (4.22),which satisfies the difference of integral identities (4.4):

$$\int_\Omega u(\boldsymbol{x}, t_0)\, \xi(\boldsymbol{x}, t_0) dx + \int_0^{t_0} \int_\Omega (-u\, \frac{\partial \xi}{\partial t} + \frac{d_0}{m} \nabla \xi \cdot\, \mathbb{C}^c \cdot \nabla\, u) dx dt = \int_0^{t_0} \int_\Omega \nabla \xi \cdot (\boldsymbol{v}\, \frac{u}{m} + \frac{u}{m^2}\, B^c \cdot \nabla\, m) dx dt,$$

where ξ is arbitrary smooth function, vanishing at $\boldsymbol{x} \in S^1 \cup S^2$ and $0 < t < t_0$ and the following homogeneous boundary and initial conditions

$$u(\boldsymbol{x}, t) = 0,\;\; \boldsymbol{x} \in S^1 \cup S^2,\;\; 0 < t < T, \quad \frac{\partial u}{\partial N} = 0,\;\; \boldsymbol{x} \in S^0,\;\; 0 < t < T,\;\; u(\boldsymbol{x}, 0) = 0,\;\; \boldsymbol{x} \in \Omega.$$

Using the standard time smoothing procedure ([25], chapter II, § 4)with the test function $\xi = (u_h)_{\bar{h}}$ and inequality $ab \leqslant \lambda\, a^2 + C_\lambda\, b^2$ we get

$$\int_\Omega (u_h)_{\bar{h}}(\boldsymbol{x}, t)\, u(\boldsymbol{x}, t) dx - \frac{1}{2} \int_\Omega u_h^2(\boldsymbol{x}, t)\, dx + \int \int_{\Omega_t} \frac{d_0}{m}\, (u_h)_{\bar{h}} \cdot (\mathbb{C}^c \cdot \nabla u) dx dt \leqslant \lambda \int \int_{\Omega_t} |(\nabla u)_{\bar{h}}|^2 dx d\tau + C_\lambda \int \int_{\Omega_t} u^2 dx d\tau.$$

The limit as $h \to 0$, strict positive definiteness of the matrix $\mathbb{C}^c$:

$$\frac{d_0}{m} \boldsymbol{\xi} \big(\cdot (\mathbb{C}^c \cdot \boldsymbol{\xi}) \big) \geqslant \nu_0\, |\boldsymbol{\xi}|^2$$

the choice is small enough λ, give us a differential inequality

$$\int_\Omega u^2(\boldsymbol{x}, t)\, dx \leqslant M^2 \int \int_{\Omega_t} u^2(\boldsymbol{x}, t)\, dx d\tau . u(\boldsymbol{x}, 0) = 0,$$

Gronwall 's lemma (Lemma 5.5, § 5,chapter II, [25]) implies $c(\boldsymbol{x}, t) = 0$ almost everywhere in Ω_T and, that is, the correct definition of the operator $\mathbb{F}^c(r)$. $\square$

5 Main result.

Theorem 4. *Let $c^0 \in \mathbb{C}^{2+\gamma}(\overline{\Omega})$, and the conditions* (4.13) *and condition*

$$0 \leqslant c^0(\boldsymbol{x}) \leqslant c^* = const \leqslant 1 \quad \boldsymbol{x} \in \Omega,\ t > 0 \tag{5.1}$$

be fulfilled.

Then for all $r \in \mathfrak{M}_T$ the problem $\mathbb{B}^\varepsilon(r)$ has a unique weak solution $\{\boldsymbol{v}^\varepsilon,\, \overline{p}^\varepsilon,\, c^\varepsilon\}$ such that the estimates

$$\|\overline{p}^\varepsilon\|_{2,\Omega^\varepsilon_{f,T}(r)} + \|\boldsymbol{v}^\varepsilon\|_{2,\Omega^\varepsilon_{f,T}(r)} + \|\sqrt{\alpha_\mu}\,\mathbb{D}(x,\boldsymbol{v}^\varepsilon)\|_{2,\Omega^\varepsilon_{f,T}(r)} \leqslant M(M_0)\,|\nabla\, p^0|^{(0)}_{\Omega} = M_p, \tag{5.2}$$

$$\|c^\varepsilon\|_{2,\Omega^\varepsilon_{f,T}(r)} + \|\nabla\, c^\varepsilon\|_{2,\Omega^\varepsilon_{f,T}(r)} \leqslant M(M_0)\,(|c^0|^{(2+\gamma)}_{\Omega} + M_p) = M_c, \tag{5.3}$$

$$\|\frac{\partial c^\varepsilon}{\partial t}\|_{W^{-1}} \leqslant M(M_c), \tag{5.4}$$

$$0 \leqslant c^\varepsilon(\boldsymbol{x},t) \leqslant c^*, \ \ (\boldsymbol{x},t) \in \Omega^\varepsilon_{f,T}(r) \tag{5.5}$$

hold true.

Here constants $M(M_p)$ and $M(M_c)$ do not depend on ε.

Theorem 5. *Let in conditions of Theorem 4 be fulfilled the Assumption* (1).

Then for all $T > 0$ the problem $\mathbb{H}(r)$ has a unique classical solution $\{\boldsymbol{v},\, p,\, c\}$ such that $\nabla p,\, \boldsymbol{v} \in \mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_T)$, $c \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$,

$$|\nabla\, p|^{(\gamma)}_{\Omega_T} + |\boldsymbol{v}|^{(\gamma)}_{\Omega_T} \leqslant M(M_p) \tag{5.6}$$

and

$$0 \leqslant c(\boldsymbol{x},t) \leqslant c^*,\ (\boldsymbol{x},t) \in \Omega_T, \tag{5.7}$$

$$|c|^{(2+\gamma)}_{\Omega_T} \leqslant M(M_c). \tag{5.8}$$

Theorem 6. *In conditions of Theorem5 for all $T > 0$ the problem $\mathbb{H}$ has a unique classical solution $\{r^*,\, \boldsymbol{v},\, p,\, c\}$, such that $r^* \in \mathfrak{M}_T$, $\nabla p,\, \boldsymbol{v} \in \mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_T)$, $c \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$ and hold true estimates* (5.6) – (5.8).

6 Proof of Theorem 4.

According to [25] (§ 9, chapter IV – correctness of the diffusion – convection problem) and [35] (Theorem 2, § 5, chapter III – correctness of the dynamic problem),

$$c^\varepsilon \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}\big(\overline{\Omega}^\varepsilon_{f,T}(r)\big),\ \boldsymbol{v}^\varepsilon,\, \nabla\, p^\varepsilon \in \mathbb{L}_\infty\Big(0,T;\mathbb{W}^2_q\big(\Omega^\varepsilon_f(r)\big)\Big) \cap \mathbb{H}^{\gamma,\frac{\gamma}{2}}\big(\overline{\Omega}^\varepsilon_{f,T}(r)\big),$$

where $q > 3(1+\gamma)$ is any.

Note that these theorems are formulated for cylindrical domains. But with the help of local estimates, all statements can be proved for non-cylindrical domains.

The belonging of the function $\boldsymbol{v}^\varepsilon$ to the space $\mathbb{H}^{\gamma,\frac{\gamma}{2}}\big(\overline{\Omega}^{\varepsilon}_{f,T}(r)\big)$ is proved in the same way as in the Lemma 7.5 (estimate (7.25)).

The dynamic problem $\mathbb{B}^\varepsilon(r)$ is a linear one. Therefore, to prove the existence of weak solutions to the problem $\mathbb{B}^\varepsilon(r)$, it is sufficient to obtain appropriate a priori estimates of the solutions to these problems.

6.1 A'priory estimates for the weak solutions to the dynamic problem $\mathbb{B}^\varepsilon(r)$

Everywhere below, for simplicity of the presentation omit the index ε where it does not cause misinterpretation. In addition, the domains $\Omega^\varepsilon_f\big(r(\boldsymbol{x},t)\big)$, $\Omega^\varepsilon_s\big(r(\boldsymbol{x},t)\big)$, $\Omega^{\boldsymbol{k},\varepsilon}_f\big(r(\boldsymbol{x},t)\big)$ and the boundary $\Gamma\big(r(\boldsymbol{x},t)\big)$ we will denote as $\Omega_f(t)$, $\Omega_s(t)$, $\Omega^{\boldsymbol{k}}_f(t)$ and $\Gamma(t)$.

Lemma 6.1. *Let $r \in \mathfrak{M}_T$.*

Then in conditions of Theorem 3 for all weak solutions to the dynamic problem $\mathbb{B}(r)$ hold true estimates (5.2).

Proof. We put $\widetilde{\boldsymbol{v}}$ in the integral identity (4.2) as a test function $\boldsymbol{\varphi}$.

One has

$$\int_{\Omega_f(t)} \alpha_\mu\, |\mathbb{D}(x,\widetilde{\boldsymbol{v}})|^2 dx = -\int_{\Omega_f(r)} (1-\chi)\sqrt{\alpha_\mu}\,\mathbb{D}\big(x,\widetilde{\boldsymbol{v}}\big) : \overset{\circ}{\mathbb{F}}\ dx \leqslant$$
$$\nu \int_{\Omega_f(t)} \alpha_\mu\, |\mathbb{D}(x,\widetilde{\boldsymbol{v}})|^2 dx + \frac{4}{\nu}\int_{\Omega_s(t)} |\overset{\circ}{\mathbb{F}}|^2 dx, \quad (6.1)$$

where $\overset{\circ}{\mathbb{F}} = \chi\sqrt{\alpha_\mu}\,\mathbb{D}(x,\overset{\circ}{\boldsymbol{v}})$ and ν is an arbitrary small positive number.

Taking $\nu = \dfrac{\mu_1}{4}$ and considering the uniform boundedness of $\overset{\circ}{\mathbb{F}}$ with respect to ε (estimate (2.14)) we will get the required estimate (5.2) for the summand $\sqrt{\alpha_\mu}\,\mathbb{D}(x,\widetilde{\boldsymbol{v}})$.

The estimate for the velocity vector follows from the Poincare inequality (part 1, § 116, [36]) for the function $\widetilde{\boldsymbol{v}}(\boldsymbol{k}\,\varepsilon + \varepsilon\boldsymbol{y},t) = \boldsymbol{u}^{\boldsymbol{k}}(\boldsymbol{y},t) = \overline{\boldsymbol{u}}^{\boldsymbol{k}}(\boldsymbol{z},t)$, which is equal to zero on the boundary of the domain $\Omega^{\boldsymbol{k},\varepsilon}_f(r)$ in the cube $\varepsilon\,Y$ with edge length ε:

$$\text{if } \int_Y |\boldsymbol{u}^{\boldsymbol{k}}|^2 dy \leqslant M^2 \int_Y |\mathbb{D}(y,\boldsymbol{u}^{\boldsymbol{k}})|^2 dy,$$
$$\text{then } \int_{\varepsilon Y} |\overline{\boldsymbol{u}}^{\boldsymbol{k}}|^2 dy \leqslant \varepsilon^2 M^2 \int_{\varepsilon Y} |\mathbb{D}(z,\overline{\boldsymbol{u}}^{\boldsymbol{k}})|^2 dy$$

with the constant M independent of ε.

Thus,

$$\int_{\Omega_f^{\boldsymbol{k},\varepsilon}(r)} |\widetilde{\boldsymbol{v}}|^2 dy \leqslant \varepsilon^2 M^2 \int_{\Omega_f^{\boldsymbol{k},\varepsilon}(r)} |\mathbb{D}(y,\widetilde{\boldsymbol{v}})^2| dy =$$
$$M^2 \int_{\Omega_f^{\boldsymbol{k},\varepsilon}(r)} \alpha_\mu |\mathbb{D}(y,\widetilde{\boldsymbol{v}})^2| dy \leqslant M^2(M_0),$$

what guarantees the estimate for the velocity vector.

To prove the boundedness of the pressure $\widetilde{p}(\boldsymbol{x},t)$ in the space $\mathbb{L}_2(\Omega_T)$ we represent the identity (4.2) as

$$l(\boldsymbol{\varphi}) \equiv \int\int_{\Omega_T} \widetilde{p}\,\nabla\cdot\boldsymbol{\varphi}\,dxdt =$$
$$\int\int_{\Omega_T} \nabla\cdot\Big(\sqrt{\alpha_\mu}\,\big((\mathbb{F}-\overset{\circ}{\mathbb{F}}) - p^0\,\mathbb{I}\big)\cdot\boldsymbol{\varphi}\,dxdt \equiv \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle, \qquad (6.2)$$

where

$$\boldsymbol{f} = \nabla\cdot\Big(\sqrt{\alpha_\mu}\,\big((\mathbb{F}-\overset{\circ}{\mathbb{F}}) - p^0\,\mathbb{I}\big),\ \ \mathbb{F} = \chi^\varepsilon\sqrt{\alpha_\mu}\,\mathbb{D}(x,\widetilde{\boldsymbol{v}}),\ \ \mathbb{F},\,\overset{\circ}{\mathbb{F}}\in\mathbb{L}_2(\Omega_T),$$
$$\boldsymbol{f}\in\mathbb{H}^{-1},\ \ \mathbb{H} = \overset{\circ}{\mathbb{W}}{}_2^{1}(\Omega_T),\ \ \mathbb{H}^{-1} = \mathbb{L}_2\big((0,T);\overset{\circ}{\mathbb{W}}{}_2^{-1}(\Omega)\big)$$

and

$$\|\boldsymbol{f}\|_{\mathbb{H}^{-1}} = \sup_{\boldsymbol{\varphi}\in\mathbb{H}} \frac{|\int\int_{\Omega_T}\Big(\sqrt{\alpha_\mu}\,\big((\mathbb{F}-\overset{\circ}{\mathbb{F}}) - p^0\,\mathbb{I}\big) : \mathbb{D}(x,\boldsymbol{\varphi})\,dxdt|}{\|\mathbb{D}(x,\boldsymbol{\varphi})\|_{2,\Omega_T}} \leqslant$$
$$\|\Big(\sqrt{\alpha_\mu}\,\big((\mathbb{F}-\overset{\circ}{\mathbb{F}}) - p^0\,\mathbb{I}\big)\|_{2,\Omega_T} \leqslant M_p. \qquad (6.3)$$

The equality (6.2) and the estimate (6.3) mean that a linear functional

$$l\,:\mathbb{L}_2(\mathbb{H}^{-1})\to\mathbb{R} \qquad (6.4)$$

is bounded (§ 1, Chapter IV, [20]) and and its norm does not exceed the value M_p, which is equivalent to the estimate (5.2) for the pressure. □

6.2 A'priory estimates for the weak solutions to the diffusion – convection problem $\mathbb{B}^\varepsilon(r)$.

Lemma 6.2. *The diffusion – convection problem* $\mathbb{B}^\varepsilon(r)$ *has an unique weak solution for which the estimates* (5.3) – (5.5) *hold true.*

Lemma 6.3. *The diffusion – convection problem* $\mathbb{B}^\varepsilon(r)$ *has an unique weak solution for which the estimates* (5.3) – (5.5) *hold true.*

Proof. To prove the lemma, we use Theorem 3 (Leray-Schauder fixed point Theorem). Let $\mathfrak{N} = \{u \in \mathbb{L}_2\big(\Omega_{f,T}(r)\big) : \|c\|_{2,\Omega_{f,T}(r)} \leqslant M(M_0)\}$.

It is easy to see that this set is closed in the metric of space $\mathbb{V}_2\big(\Omega_{f,T}(r)\big)$.

Next we define an operator $s^\tau = \boldsymbol{\Psi}^\tau(u)$, s^τ is the solution to the modified diffusion – convection problem $\mathbb{B}(r)$, where the function $\psi(c)$ is replaced by the function $\tau\,\psi(u)$.

Let's call this linear problem as a problem $\mathbb{B}^\tau(r,u)$.

By construction the solution s^τ to the problem $\mathbb{B}^\tau(r,u)$ satisfies an integral identity

$$\int_{\Omega_f(t_0)} s^\tau(\boldsymbol{x},t_0)\,\xi(\boldsymbol{x},t_0)dx - \int_{\Omega(0)} s^\tau(\boldsymbol{x},0)\,\xi(\boldsymbol{x},0)dx + \\ \int_0^{t_0}\int_{\Omega_f(t)} \big(- s^\tau\,\frac{\partial \xi}{\partial t} + \nabla\xi\cdot\big(d_0\,\nabla\, s^\tau - \tau\,\widetilde{\boldsymbol{v}}\,\psi(u)\big)dxdt = 0, \quad (6.5)$$

which holds true for all t_0, $0 \leqslant t_0 \leqslant T$ and for all smooth functions ξ, vanishing at the boundary $S^1 \cup S^2$ for $t \geqslant 0$.

The linear problem $\mathbb{B}^\tau(r,u)$ has an unique weak solution s^τ from the space $\mathbb{V}_2\big(\Omega_{f,T}(r)\big)$and estimates (5.3) and (5.4) are fair for its solutions. Due to the simplicity of this statement, we were unable to find publications with proof of the formulated statement, but its solvability is simple enough to prove using the Galerkin method and the methods developed in the monograph [25].

Estimates (5.3) and (5.4) for s^τ and Theorem 2 show, that the operator $\boldsymbol{\Psi}^\tau(u)$ is completely continuous and maps the set $\mathfrak{N}$ into itself.

For $\tau = 0$ the problem $\mathbb{B}^0(r,u)$ has a solution s^0. Therefore, the operator $\boldsymbol{\Psi}^\tau(u)$ has at least one fixed point $s^1 = c$ which will be the weak solution to the modified diffusion – convection problem $\mathbb{B}^1(r,c)$.

Only now that can we prove the estimate (5.5) for the solutions c to the modified diffusion – convection problem $\mathbb{B}(r)$.

We put in the identity (4.6) $c = u + c^*$ and $\xi = u^+ = \max\{u, 0\}$.

By definition $u^+ \geqslant 0$, $u^+\,u = |u^+|^2$, $\nabla\,u\cdot\nabla\,u^+ = |\nabla\,u^+|^2$, $u^+ = 0$ at the boundary $S^1 \cup S^2$ and $\dfrac{\partial u^+}{\partial t}\,u = \dfrac{\partial u}{\partial t}\,u^+$.

Taking into account the formulas (4.5) we arrive at a chain of the equal-

ities:

$$\int_{\Omega_f(t_0)} \big(c^* + u(\boldsymbol{x},t_0)\big)\, u^+(\boldsymbol{x},t_0)dx = \int_{\Omega_f(t_0)} \big(c^* u^+(\boldsymbol{x},t_0) + |u(\boldsymbol{x},t_0)|^2\big)\, dx =$$

$$\int_0^{t_0}\int_{\Omega_f(t)} (u+c^*)\frac{\partial u^+}{\partial t} dxdt - I_1 + I_2 =$$

$$\int_0^{t_0}\int_{\Omega_f(t)} \frac{\partial}{\partial t}(c^*\, u^+ + \frac{1}{2}\,|u^+|^2) dxdt + I_1 - I_2 =$$

$$\int_{\Omega_f(t_0)} \big(c^* u^+(\boldsymbol{x},t_0) + \frac{1}{2}\,|u^+(\boldsymbol{x},t_0)|^2\big) dx + I_\Gamma + I_1 - I_2.$$

$$I_1 = d_0 \int_0^{t_0}\int_{\Omega_f(t)} \nabla\, u^+ \cdot \nabla\, u\, dxdt = d_0 \int_0^{t_0}\int_{\Omega_f(t)} |\nabla\, u^+|^2\, dxdt,$$

$$I_2 = \int_0^{t_0}\int_{\Omega_f(t)} \nabla\, u^+ \cdot \widetilde{\boldsymbol{v}}\, \psi(u^+ + c^*) dxdt,\ \ I_\Gamma = \int_0^{t_0}\int_{\Gamma(t)} (c^* u^+ + \frac{1}{2}\,|u^+|^2)\, D_N^\varepsilon\, \sin\varphi\, d\sigma dt.$$

Since $D_N^\varepsilon\, \sin\varphi \geqslant 0$, then we finally get the inequality

$$\frac{1}{2}\int_{\Omega_f(t_0)} |u^+(\boldsymbol{x},t_0)|^2 dx \leqslant -\int_0^{t_0}\int_{\Omega_f(t)} \nabla\, u^+ \cdot \widetilde{\boldsymbol{v}}\, \psi(u+c^*) dxdt, \tag{6.6}$$

from which it follows that $u^+(\boldsymbol{x},t) = 0$ almost everywhere in $\Omega_{f,T}$.

In fact, otherwise on the set Q_u where $u > 0$ (or where $c > c^*$) the left hand – side of the inequality (6.6) strictly positive, whereas the right hand –side of this inequality is zero. That is$c \leqslant c^*$. The case $c \geqslant 0$ is considered similarly.

Estimates (5.5) the solutions to the modified diffusion – convection problem $\mathbb{B}^\varepsilon(r)$ show, that in the identity (4.6) $\psi(s) = s + \dfrac{\theta}{\delta}$. That is, this problem coincides with the diffusion – convection problem $\mathbb{B}^\varepsilon(r)$ and for extended functions $\widetilde{\boldsymbol{v}}^\varepsilon$ and $\widetilde{c}^\varepsilon$ the identity

$$\int_\Omega \chi(r,\frac{\boldsymbol{x}}{\varepsilon})\, \widetilde{c}^\varepsilon(\boldsymbol{x},t_0)\, \chi\big(r(\boldsymbol{x},t_0),\frac{\boldsymbol{x}}{\varepsilon}\big) dx - \int_\Omega \chi\big(r(\boldsymbol{x},0),\frac{\boldsymbol{x}}{\varepsilon}\big)\, c^0(\boldsymbol{x})\, \xi(\boldsymbol{x},0) dx +$$

$$\int_0^{t_0}\int_\Omega \chi(r,\frac{\boldsymbol{x}}{\varepsilon})\Big(- \widetilde{c}^\varepsilon \frac{\partial\xi}{\partial\tau} + \nabla\xi\cdot\big(d_0\, \nabla\, \widetilde{c}^\varepsilon - \widetilde{\boldsymbol{v}}^{\,\varepsilon}\,(\widetilde{c}^\varepsilon + \frac{\theta}{\delta})\big)\Big) dxd\tau = 0 \tag{6.7}$$

fulfilled for all smooth functions ξ, vanishing at the boundary $S^1 \cup S^2$ of the domain Ω for $t > 0$.

Remark 3. In particulary, the identity (6.7) means that
$\dfrac{\partial \widetilde{c}^\varepsilon}{\partial t} \in \mathbb{L}_2\big((0,T);\mathbb{W}_2^{-1}(\Omega)\big)$ and

$$\|\frac{\partial \widetilde{c}^\varepsilon}{\partial t}\|_{W_2^{-1}} \leqslant M(M_c). \tag{6.8}$$

$\square$

7 Proof of Theorem 5.

7.1 Homogenization: the choice of convergent subsequences.

Estimates (5.2) – (5.4) allow you to select converging sequences as $\varepsilon \to 0$ (for simplicity, we will leave the same indexes)

$$\begin{gathered}
\widetilde{\boldsymbol{v}}^{\varepsilon} \rightharpoonup \boldsymbol{v},\ \widetilde{\boldsymbol{v}}^{\varepsilon} \stackrel{t.-s.}{\rightharpoonup} \boldsymbol{V},\ (1-\chi^{\varepsilon})\widetilde{\boldsymbol{v}}^{\varepsilon} \text{ converges strongly in } \mathbb{L}_2(\Omega_T) \text{ to zero;}\\
\varepsilon\,\chi^{\varepsilon}\,\mathbb{D}(x,\widetilde{\boldsymbol{v}}^{\varepsilon}) \stackrel{t.-s.}{\rightharpoonup} \mathbb{D}\big(y,\boldsymbol{V}\big),\ \bar{p}^{\varepsilon} \rightharpoonup p,\ \widetilde{p}^{\varepsilon} \stackrel{t.-s.}{\rightharpoonup} P;\\
\sqrt{\alpha_\mu}\ \overset{\circ}{\mathbb{F}}{}^{\varepsilon} \text{ converges strongly in } \mathbb{L}_2(\Omega_T) \text{ to zero;}\\
\overset{\circ}{\boldsymbol{v}}{}^{\varepsilon} \text{ converges strongly in } \mathbb{L}_2(\Omega_T) \text{ to zero;}\\
\widetilde{c}^{\varepsilon} \text{converges strongly in } \mathbb{L}_2(\Omega_T) \text{ to } c,\ \chi^{\varepsilon}\,\widetilde{c}^{\varepsilon} \stackrel{t.-s.}{\rightharpoonup} m\,c;\\
\nabla\,\widetilde{c}^{\varepsilon} \stackrel{t.-s.}{\rightharpoonup} \nabla_x\,c+\nabla_y\,C,
\end{gathered} \tag{7.1}$$

where functions $\boldsymbol{V}=\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{P}=\boldsymbol{P}(\boldsymbol{x},t,\boldsymbol{y})$ and $\boldsymbol{C}=\boldsymbol{C}(\boldsymbol{x},t,\boldsymbol{y})$ are 1 – periodic in the variable $\boldsymbol{y}$,

$$\boldsymbol{v},\ p \in \mathbb{L}_2(\Omega_T),\ c \in \mathbb{W}^{1,0}_2(\Omega_T),\ \ P,\ \boldsymbol{V},\ C,\ \mathbb{D}\big(y,\boldsymbol{V}\big),\ \nabla_y\,C \in \mathbb{L}_2(,\Omega_T\times Y)$$

and

$$\begin{gathered}
\|\boldsymbol{v}\|_{2,\Omega_T}+\|p\|_{2,\Omega_T} \leqslant M_p,\\
\|P\|_{2,Y\times\Omega_T}+\|\boldsymbol{V}\|_{2,Y\times\Omega_T}+\|\mathbb{D}\big(y,\boldsymbol{V}\big)\|_{2,Y\times\Omega_T} \leqslant M_p;\\
0 \leqslant c(\boldsymbol{x},t) \leqslant c^*,\ (\boldsymbol{x},t)\in\Omega_T,\\
\|\nabla\,c\|_{2,\Omega_T}+\|\nabla\,C\|_{2,Y\times\Omega_T} \leqslant M_c.
\end{gathered} \tag{7.2}$$

7.2 Homogenization of the problem $\mathbb{B}^{\varepsilon}(r)$

Lemma 7.1. *In conditions of the Theorem 4 the macroscopic continuity equation*

$$\nabla\cdot\boldsymbol{v}=\delta\,\frac{\partial m}{\partial t} \tag{7.3}$$

and the first boundary condition in (4.16) *for the velocity vector hold true in the domain* Ω_T.

In (7.3)

$$m(r)=\int_Y \chi(\boldsymbol{x},t,\boldsymbol{y})dy=1-\frac{4}{3}\,\pi\,r^3.$$

Proof. The limit as $\varepsilon \to 0$ in the identity (4.3) with smooth test functions ψ, vanishing at the boundary $S^1 \cup S^2$ of the domain Ω for $t > 0$, gives us the integral identity

$$0 = \int\int_{\Omega_T} (-\delta\, m \,\frac{\partial \psi}{\partial t} + \boldsymbol{v} \cdot \nabla \psi)\, dxdt = \int\int_{\Omega_T} (\delta \,\frac{\partial m}{\partial t}\, \psi + \boldsymbol{v} \cdot \nabla \psi)\, dxdt. \quad (7.4)$$

The identity (7.4) means that differential equation (7.3) and the first boundary condition in (4.16) are satisfied in a usual sense. □

Lemma 7.2. *In conditions of the Theorem 4 the limiting functions* $\boldsymbol{v}$ *and* p *are bounded in the spaces* $\mathbb{L}_2(\Omega_T)$ *and* $\mathbb{W}_2^{1,0}(\Omega_T)$ *correspondingly:*

$$\|\boldsymbol{v}\|_{2,\Omega_T} + \|\nabla\, p\|_{2,\Omega_T} \leqslant M_p, \quad (7.5)$$

and satisfy the **Darcy 's law of filtration**

$$\boldsymbol{v} = -\frac{1}{\mu_1}\, \mathbb{C}^v \cdot \nabla\, (p + p^0) \quad (7.6)$$

in the domain Ω_T *and the second boundary condition* (4.16)*.*

A symmetric strictly positive definite matrix $\mathbb{C}^v(r)$ *is given by the formulas* (4.19)*.*

Proof. First of all we prove that

$$P(\boldsymbol{x}, t, \boldsymbol{y}) = p(\boldsymbol{x}, t)\, \chi(\boldsymbol{x}, t, \boldsymbol{y}). \quad (7.7)$$

To do that we put in the identity (4.2) $\boldsymbol{\varphi} = \varepsilon\, \varphi_0(\boldsymbol{x}, t)\, \boldsymbol{\varphi}_1(\frac{\boldsymbol{x}}{\varepsilon})$, where $\varphi_0(\boldsymbol{x}, t)$ is an arbitrary smooth function in the domain Ω_T, vanishing at the lateral boundary of the domain Ω_T, and $\boldsymbol{\varphi}_1(\boldsymbol{y})$ arbitrary smooth function in Y.

Passing to the limit as $\varepsilon \to 0$ we get

$$\int\int_{\Omega_T} \varphi_0(\boldsymbol{x}, t\big(\int_Y P(\boldsymbol{x}, t, \boldsymbol{y})\, \nabla_y \cdot \boldsymbol{\varphi}_1(\boldsymbol{y}) dy\big) dxdt = 0,$$

which is equivalent to equality (7.7).

The continuity equation and the boundary condition for the velocity vector $\boldsymbol{V}$:

$$\nabla_y \cdot \boldsymbol{V} = 0,\ \ \boldsymbol{y} \in Y_f(r),\ \ \boldsymbol{V} = 0,\ \ |\boldsymbol{y}| = r,\ \ 0 < t < T \quad (7.8)$$

are derived in [15].

To derive continuity equation we take as test functions ψ in the integral identity (4.3) functions of the type $\psi = \varepsilon\, \psi_0(\boldsymbol{x}, t)\, \psi_1(\frac{\boldsymbol{x}}{\varepsilon})$ and pass to the limit as $\varepsilon \to 0$.

To derive boundary condition in (7.8) we consider a two – scale limit in the equality $(1-\chi^{\varepsilon})\,\widetilde{\boldsymbol{v}}^{\varepsilon}=0$, which leads to the relation $\big(1-\chi(\boldsymbol{x},t,\boldsymbol{y})\big)\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})=0$.

Next we use the smoothness of the function $\boldsymbol{V}$: $\boldsymbol{V}\in\mathbb{W}_2^{1,0}(Y)$, which dictates the boundary condition in (7.8).

The limit as $\varepsilon\to 0$ in (4.2) gives us the ***Homogenized dynamics equation***

$$\int_0^T\int_\Omega\Big(\int_{Y_f(r)}\big(\mu_1\,\mathbb{D}(y,\boldsymbol{V}):\mathbb{D}(y,\boldsymbol{\varphi})-p\big)\,\nabla\cdot\boldsymbol{\varphi}\,dy\Big)dxdt=$$
$$-\int_0^T\int_\Omega\nabla\,p^0\cdot\boldsymbol{\varphi}\,dxdt. \quad (7.9)$$

Now we put in (7.9) $\boldsymbol{\varphi}=\varphi_0(\boldsymbol{x},t)\,\boldsymbol{\varphi}_i(\dfrac{\boldsymbol{x}}{\varepsilon})$, where solenoidal functions $\boldsymbol{\varphi}_i(\boldsymbol{y})$ are equal to zero at the boundary $\gamma(r)$ of the liquid component $Y_f(r)$ of the domain Y, $\boldsymbol{\varphi}_i\in\mathbb{W}_2^1(Y_f(r))$, $\operatorname{supp}\boldsymbol{\varphi}_i\in Y_f(r)$, $\varphi_i(\boldsymbol{x},t)=0$ at the boundary S^0 of the domain Ω for $t>0$, supp $\boldsymbol{\varphi}_i\subset Y_f(r)$ and $\langle\boldsymbol{\varphi}_i\rangle_{Y_f}=\boldsymbol{e}_i$, where $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is orthonormal basis in $\mathbb{R}^3$. Such a choice is possible due to the Lemma 2.1.

One has

$$\int\int_{\Omega_T}\big(\varphi_0\,a_i-p\,\frac{\partial\varphi_0}{\partial x_i}\big)dxdt=-\int_0^T\int_\Omega\nabla\,p^0\,dxdt, \quad (7.10)$$

$$a_i=\int_{Y_f}\mu_1\,\mathbb{D}(y,\boldsymbol{V}):\mathbb{D}(y,\boldsymbol{\varphi}_{1,i})dy\in\mathbb{L}_2(\Omega_T).$$

Therefore, $a_i=-\dfrac{\partial p}{\partial x_i}$ and

$$\|\nabla\,p\|_{2,\Omega_T}\leqslant M\sum_{i=1}^3\|\int_{Y_f}\mu_1\,\mathbb{D}\big(y,\boldsymbol{V}\big):\mathbb{D}(y,\boldsymbol{\varphi}_{1,i})dy\|_{2,\Omega_T}\leqslant M_p,$$

which is proves the estimate (7.5) for the pressure.

Since in the identity (4.2) functions $\varphi_0$are equal to zero only at the part of the boundary S^0 of the boundary S for $t>0$, then p is obliged be zero at the part $S^1\cup S^2$ of the boundary S for $t>0$, that is proves the second boundary condition in for the pressure in (4.16).

Since the pressure belongs to the space $\mathbb{W}_2^{1,0}(\Omega_T)$, then equalities (7.8) might be rewritten in the form of the Stokes equations

$$\mu_1\,\triangle_y\,\boldsymbol{V}-\nabla_y\,\Pi=\nabla\,(p+p^0),\ \ \nabla_y\cdot\boldsymbol{V}=0,\ \ \boldsymbol{y}\in Y_f;$$
$$\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})=0,\ \ \boldsymbol{y}\in\gamma(r).$$

The solution of the last system is given by an explicit formula

$$\boldsymbol{V} = -\frac{1}{\mu_1}\sum_{i=1}^{3}(\boldsymbol{V}^i \otimes \boldsymbol{e}_i)\cdot \nabla\,(p+p^0),$$

where functions $\boldsymbol{V}^i(\boldsymbol{y})$ are defined by formulas (4.19).

Thus,

$$\boldsymbol{v} = \langle \boldsymbol{V}\rangle_{Y_f} = -\frac{1}{\mu_1}\langle\sum_{i=1}^{3}(\boldsymbol{V}^i \otimes \boldsymbol{e}_i)\rangle_{Y_f}\cdot \nabla\,(p+p^0) = -\frac{1}{\mu_1}\,\mathbb{C}^v\cdot \nabla\,(p+p^0),$$

which is complete the prove of the Lemma. □

Consequence 7.1. The limiting functions $\boldsymbol{v}(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t)$ are defined uniquely.

The proof of the last statement follows from the estimate (7.2) for the difference of two possible solutions to the system of Darcy's system of differential equations (7.3), (7.6), completed with boundary conditions (4.16).

7.3 Homogenization of the diffusion – convection problem $\mathbb{B}^{\varepsilon}(r)$.

Lemma 7.3. *In conditions of the Theorem 4 the diffusion – convection problem $\mathbb{H}(r)$ has a unique weak solution $c \in \mathbb{V}_2(\Omega_T)$ such that*

$$\|c\|_{2,\Omega_T} + \|\nabla\, c\|_{2,\Omega_T} \leqslant M_c,\ \ 0 \leqslant c(\boldsymbol{x},t) \leqslant c^*,\ \ (\boldsymbol{x},t)\in\Omega_T \tag{7.11}$$

and the integral identity

$$\int_{\Omega} m\,c(\boldsymbol{x},t_0)\,\xi(\boldsymbol{x},t_0)dx - \int_{\Omega} m\,c^0(\boldsymbol{x})\,\xi(\boldsymbol{x},0)dx + \int_0^{t_0}\int_{\Omega}\Big(-m\,c\frac{\partial\xi}{\partial t} + \nabla\xi\cdot\big(d_0\,\mathbb{C}^c(r)\nabla\,c - \boldsymbol{v}\,(c+\frac{\theta}{\delta})\big)\Big)dxdt = 0, \tag{7.12}$$

is fulfilled for all smooth functions ξ vanishing at the boundary $S^1\cup S^2$ of the domain Ω for $t\geqslant 0$.

Symmetric strictly positive definite matrix $\mathbb{C}^c(r)$ is defined by formula (4.19).

To prove identity (7.12) it is enough to pass to the limit as $\varepsilon\to 0$ in the identity (4.4).

Estimates (7.11) follow from estimates (5.3) and (5.5).

Properties of the matrices $\mathbb{C}^v$ and $\mathbb{C}^c$ have been studied in [15] for the case of the fixed structure of the pore space.

In our situation these matrices will only be positively definite for all $r(\boldsymbol{x},t)>0$. That is, the result we need is a uniform evaluation from below of the eigenvalues of the matrices $\mathbb{C}^v$ $\mathbb{C}^c$, which is valid for all $r(\boldsymbol{x},t)>0$. Therefore, we have to prove additionally the desired statement.

Lemma 7.4. *In conditions of the Theorem 4 symmetric strictly positive definite matrices $\mathbb{C}^v(r)$ and $\mathbb{C}^c(r)$ are infinitely smooth with respect to parameter r:*

$$\big(\mathbb{C}^v(r)\cdot\boldsymbol{\zeta}\big)\cdot\boldsymbol{\zeta}+\big(\mathbb{C}^c(r)\cdot\boldsymbol{\zeta}\big)\cdot\boldsymbol{\zeta}\geqslant\nu_0>0 \tag{7.13}$$

with strictly positive constant ν_0 for all $\boldsymbol{\zeta}\in\mathbb{R}^3$, $|\boldsymbol{\zeta}|=1$.

Proof. First of all we will consider the matrix $\mathbb{C}^v(r)$, prove the solvability of the problem (4.19) for $r(\boldsymbol{x},t)=0$ and study the properties of its solutions $\boldsymbol{V}^i(r;\boldsymbol{y})$ for all $r(\boldsymbol{x},t)\geqslant 0$.

Recall,that matrix $\mathbb{C}^v(r)$ is defined by solutions $\boldsymbol{V}^i(r;\boldsymbol{y})$ to the boundary – value problem (4.19)

$$\begin{aligned}\triangle_y\boldsymbol{V}^i-\nabla_y\Pi^i=-\boldsymbol{e}^i,\ \nabla_y\cdot\boldsymbol{V}^i=0,\quad |\boldsymbol{y}|>r,&\\ \boldsymbol{V}^i(\boldsymbol{y})=0,\quad |\boldsymbol{y}|=r,\ \int_{Y_f}\Pi^i(\boldsymbol{y})dy=0&\end{aligned} \tag{7.14}$$

as

$$\mathbb{C}^v(r)=2\sum_{i=1}^{3}\langle\boldsymbol{V}^i\rangle_{Y_f}\otimes\boldsymbol{e}^i.$$

It is easy to see that elements of the matrix $\mathbb{C}^v(r)$ are infinitely smooth with respect to parameter r.

Multiplication of the dynamic Stokes equation in (7.14) by $\boldsymbol{V}^j$ and integration by parts over domain $Y_f(r)$ give us

$$\langle\mathbb{D}\big(y,\boldsymbol{V}^j\big):\mathbb{D}\big(y,\boldsymbol{V}^i\rangle_{Y_f}=\langle\boldsymbol{V}^j\cdot\boldsymbol{e}^i\rangle_{Y_f},\ \ i,j=1,2,3. \tag{7.15}$$

Relations (7.15) show, that the matrix $\mathbb{C}^v(r)$ is symmetric.

These equalities for $i=j$ and the Poincare - Friedrichs inequality (§ 116, chapter III, Vol. 4, [36]) imply the estimate

$$\begin{aligned}&\int_{Y_f(r)}|\mathbb{D}\big(y,\boldsymbol{V}^i\big)|^2dy=|\big(\int_{Y_f(r)}\boldsymbol{V}^i_r dy\big)\cdot\boldsymbol{e}^i|\leqslant\\&\qquad\nu\int_{Y_f(r)}|\boldsymbol{V}^i|^2dy+\frac{1}{4\,\nu}\,m_r\leqslant\nu\,M_3\int_{Y_f(r)}|\mathbb{D}\big(y,\boldsymbol{V}^i\big)|^2dy+\frac{1}{4\,\nu}\,m(r),\end{aligned}$$

for an arbitrary positive ν with some constant M_3 in the Poincare - Friedrichs inequality independent of r.

Taking $\nu\, M_3=\dfrac{1}{2}$ we obtain

$$\int_{Y_f(r)}\big(|\boldsymbol{V}^i|^2+|\mathbb{D}\big(y,\boldsymbol{V}^i\big)|^2\big)dy\ \leqslant\ M_3\, m(r). \tag{7.16}$$

Functions $\boldsymbol{V}^i$ and $\mathbb{D}\big(y,\boldsymbol{V}^i\big)$ are continuous with respect to the parameter r and

$$\lim_{r\to 0}\boldsymbol{V}^i=\boldsymbol{V}^i_0,\ \ \lim_{r\to 0}\mathbb{D}\big(y,\boldsymbol{V}^i\big)=\mathbb{D}\big(y,\boldsymbol{V}^i_0\big),$$

where

$$\triangle_y\boldsymbol{V}^i_0-\nabla_y\Pi^i_0=-\boldsymbol{e}_i,\ \ \nabla_y\cdot\boldsymbol{V}^i_0=0,\quad |\boldsymbol{y}|>0,$$
$$\boldsymbol{V}^i_0(r;\boldsymbol{0})=0,\quad \int_Y\Pi^i_0(r;\boldsymbol{y})dy=0 \tag{7.17}$$

Let now consider arbitrary constant vectors $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\zeta_3)$ and $\boldsymbol{\eta}=(\eta_1,\eta_2,\eta_3)$ from $\mathbb{R}^3$ with unit norms and put

$$\boldsymbol{z}_\zeta(\boldsymbol{y})=\sum_{i=1}^3\zeta_i\boldsymbol{V}^i(\boldsymbol{y}),\ \ \boldsymbol{z}_\eta(\boldsymbol{y})=\sum_{j=1}^3\eta_j\boldsymbol{V}^j(\boldsymbol{y}),$$

$$f(\boldsymbol{\eta})=\big(\mathbb{C}^v(r)\cdot\boldsymbol{\eta}\big)\cdot\boldsymbol{\eta}=\int_{Y_f(r)}|\mathbb{D}(y,\boldsymbol{z}_\eta)|^2dy.$$

The continuous function $f(\boldsymbol{\eta})$ reaches its minimum $f^0=f(\boldsymbol{\eta^0})$ on the sphere $|\boldsymbol{\eta}|=1$.

Suppose that $f^0=0$. Then, $\mathbb{D}(y,\boldsymbol{z}_{\eta^0})=0$, which which is possible only if $\boldsymbol{z}_{\eta^0}$ is a linear function of the variable $\boldsymbol{y}$.

On the other hand – side the boundary – value problem (7.17) implies

$$\triangle_y\boldsymbol{z}_{\eta^0}-\nabla_y\,\Pi_{\eta^0}=-\boldsymbol{\eta^0},\quad \nabla_y\cdot\boldsymbol{z}_{\eta^0}=0,\quad |\boldsymbol{y}|>0,$$
$$\boldsymbol{z}_{\eta^0}(\boldsymbol{0})=0,\quad \int_Y\Pi_{\eta^0}(\boldsymbol{y})dy=0.$$

Therefore $\Pi_{\eta^0}=\boldsymbol{\eta}^0\cdot\boldsymbol{y}$ everywhere in the cube Y. Since the function Π_{η^0} is periodic in $\boldsymbol{y}$, it is possible only if $\boldsymbol{\eta^0}=0$.

The obtained contradiction proves our statement.

Let us now derive formulas (4.19) for the functions $C^i(\boldsymbol{y})$.

First we consider the limiting identity, resulting from identity (4.4) as $\varepsilon\to 0$ with test functions $\xi=\xi(\boldsymbol{x},t)$:

$$0=\int_0^{t_0}\int_\Omega\Big(-m\,c\,\frac{\partial\xi}{\partial t}+d_0\,\nabla\,\xi\cdot\big(\nabla\,c+\int_{Y_f(r)}\nabla_y\,C\,dy-\boldsymbol{v}(c+\frac{\theta}{\delta})\big)\Big)dxdt=$$
$$\int\int_{\Omega_T}\big(-m\,c\,\frac{\partial\xi}{\partial t}+d_0\,\nabla\,\xi\,\cdot(\nabla\,c+\int_{Y_f(r)}\nabla_y\,C\,dy+\boldsymbol{f})\big)dxdt=$$
$$\int_0^{t_0}\int_\Omega\Big(-m\,c\,\frac{\partial\xi}{\partial t}+d_0\,\nabla\,\xi\,\cdot\big(\mathbb{C}^c(r)\cdot\nabla\,c+\boldsymbol{f}\big)\Big)dxdt. \tag{7.18}$$

Here

$$d_0\,\boldsymbol{f}(\boldsymbol{x},t) = -(\boldsymbol{v} + \frac{\theta}{\delta}) = (f_1, f_2, f_3),$$

$$\mathbb{C}^c(r) = \mathbb{I} + \int_{Y_f(r)} \nabla_y\, C\, dy = \mathbb{I} + \mathbb{C}_0^c(r).$$

Next as test functions we choose functions $\xi = \varepsilon\,\xi_0(\boldsymbol{x},t)\,\xi_1(\frac{\boldsymbol{x}}{\varepsilon})$.

The limit as $\varepsilon \to 0$ gives us identity

$$\int_0^{t_0} \int_\Omega \xi_0\big(\int_{Y_f} \nabla_y \xi_1 \cdot (\nabla_y\, C + \boldsymbol{f}) dy\big) dx dt = 0,$$

where from

$$\nabla_y\, \cdot (\nabla_y C + \boldsymbol{f}) = 0, \quad \boldsymbol{y} \in Y_f, \quad \frac{\partial C}{\partial n} + \boldsymbol{f} \cdot \boldsymbol{n} = 0,\ |\boldsymbol{y}| = r \tag{7.19}$$

with $\boldsymbol{n} = \dfrac{\boldsymbol{y}}{r}$.

As usual, we use the representation

$$C(r;\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^{3} C^i(r;\boldsymbol{y})\, f_i(\boldsymbol{x},t).$$

One has

$$\nabla_y\, \cdot (\nabla_y C^i + \boldsymbol{e}^i) = 0, \quad \boldsymbol{y} \in Y_f, \quad \frac{\partial C^i}{\partial n} + \boldsymbol{e}^i \cdot \boldsymbol{n} = 0,\ |\boldsymbol{y}| = r. \tag{7.20}$$

Multiplying the differential equation for C^i in (7.20) by C^j and integrating by parts over domain $Y_f(r)$ we arrive at the equalities

$$\int_{Y_f(r)} \nabla_y\, C^i \cdot \nabla_y\, C^j\, dy + \int_{Y_f(r)} \nabla_y\, C^j \cdot \boldsymbol{e}^i\, dy = 0,\ \ i,j, = 1,2,3. \tag{7.21}$$

In particular,

$$\int_{Y_f(r)} |\nabla_y\, C^i|^2\, dy + \int_{Y_f(r)} \nabla_y\, C^i \cdot \boldsymbol{e}^i\, dy = 0,\ \ i, = 1,2,3,$$

whence follows the a'priori estimate

$$\sum_{i=1}^{3} \int_{Y_f(r)} |\nabla_y\, C^i|^2\, dy = |\sum_{i=1}^{3} \int_{Y_f(r)} \nabla_y\, C^i \cdot \boldsymbol{e}^i\, dy| \leqslant$$

$$\frac{1}{2} \sum_{i=1}^{3} \int_{Y_f(r)} |\nabla_y\, C^i|^2\, dy + \frac{3}{2}\, m(r);$$

$$\sum_{i=1}^{3} \int_{Y_f(r)} |\nabla_y\, C^i|^2\, dy \leqslant\ M_4\, m(r),$$

with the constant M_4 independent of the choice of the function $r \in \mathfrak{M}_T$, and unambiguous solvability of the boundary value problem (7.20) in the space $\mathbb{W}_2^1(Y_f(r))$ for all $r > 0$.

Besides, from (7.20) follow equalities

$$\big(\mathbb{C}_0^c(r) \cdot \boldsymbol{e}^i\big) \cdot \boldsymbol{e}^j + \int_{Y_f(r)} \nabla_y C^i \cdot \nabla_y C^j \, dy = 0, \;\; i, j, = 1, 2, 3, \tag{7.22}$$

That mean the symmetry of the matrix $\mathbb{C}_0^c(r)$ and hence the matrix $\mathbb{C}^c(r)$.

Strict positive definiteness of the matrix $\mathbb{C}^c(r)$ is proved completely analogous to the strict positive definiteness of the matrix $\mathbb{C}^v(r)$, which completes the proof of the Lemma. $\square$

Consequence 7.2. We can always assume that the matrices $\mathbb{C}^v(r)$ $\mathbb{C}^c(r)$ are diagonal.

To prove this fact, it is enough to refer to the Theorem 5.28 (§ 6, Chapter 5, [38])on the reduction of two symmetric matrices, one of which is positive definite, to a diagonal form by means of an orthogonal transformation of the coordinate system.

In this case, a positive definite matrix is reduced to a unit matrix.

Everywhere below we will assume that our Cartesian coordinate system $\{\boldsymbol{e}_1, \, \boldsymbol{e}_2, \, \boldsymbol{e}_3\}$ is an orthonormal system of eigenvectors of the matrix $\mathbb{C}^v(r) = \mathbb{I}$ and $\mathbb{C}^c(r) = \sum\limits_{i=1}^{3} c^i(r) \, \boldsymbol{e}_i \otimes \boldsymbol{e}_i$. Moreover, due to the symmetry of the set Y_f regarding orthogonal transformations

$$c^1(r) = c^2(r) = c^3(r) = \frac{1}{d_0} \, s(r) \geqslant \, s_0 = \mathrm{const} > 0.$$

Consequence 7.3. The solution $\{\boldsymbol{v}, \, p, \, c\}$ to the problem $\mathbb{H}(r)$ satisfies the following initial boundary – value problem

$$\boldsymbol{v} = -\frac{1}{\mu_1} \nabla \, (p + p^0), \;\; \nabla \cdot \boldsymbol{v} = \delta \, \frac{\partial m}{\partial t}, \;\; (\boldsymbol{x}, t) \in \Omega_T,$$

$$\frac{\partial p}{\partial n} = 0, \;\; (\boldsymbol{x}, t) \in S_T^0, \;\; p = 0, \;\; (\boldsymbol{x}, t) \in S_T^1 \cup S_T^2; \tag{7.23}$$

$$\frac{\partial}{\partial t}(m \, c) = \nabla \cdot \big(s(r) \cdot \nabla \, c - \boldsymbol{v} \, (c + \frac{\theta}{\delta})\big), \;\; (\boldsymbol{x}, t) \in \Omega_T,$$

$$c = c^0, \;\; (\boldsymbol{x}, t) \in S_T^1 \cup S_T^2,$$

$$\frac{\partial c}{\partial n} = 0, \;\; (\boldsymbol{x}, t) \in S_T^0, \;\; c(\boldsymbol{x}, 0) = c^0(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega, \tag{7.24}$$

where $\boldsymbol{n}$ is the unit normal vector to the boundary S_T^0.

The initial boundary – value problem for the limiting concentration c is understood in the sense of distributions as the integral identity (4.17), completed with the boundary condition (7.24) at the boundary S_T^1.

7.4 Differential properties of solutions to the problem $\mathbb{H}(r)$.

Lemma 7.5. *In conditions of the Theorem 4 the solutions $\boldsymbol{v}$ and ∇p to the problem (7.23) are from $\mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_T)$ and the following a'priori estimate*

$$|\nabla p|^{(\gamma)}_{\Omega_T} + |\boldsymbol{v}|^{(\gamma)}_{\Omega_T} \leqslant M(M_p) \tag{7.25}$$

hold true

Proof. The function $p(\boldsymbol{x},t)$ satisfies in the domain Ω for $t>0$ the following boundary – value problem

$$\begin{gathered}\triangle p = f, \;\; f = -\mu_1\,\delta\,\frac{\partial m}{\partial t}, \\ p(\boldsymbol{x},t)=0, \;\;\; \boldsymbol{x}\in S^1\cup S^2, \;\; t>0, \;\; \frac{\partial p}{\partial N}(\boldsymbol{x},t)=0, \;\; \boldsymbol{x}\in S^0, \;\; t>0; \\ f(.,t)\in \mathbb{H}^{\frac{1+\gamma}{2}}[0,T].\end{gathered} \tag{7.26}$$

Let $\widetilde{p}(\boldsymbol{x},t)$ be the extension of $p(\boldsymbol{x},t)$ across the boundary $S=\partial\Omega$ into the domain $Q=\{\boldsymbol{x}\in\mathbb{R}^3 : |x_i|<1, \, i=1,2,3\}$ in the direction of the normal vector to S in an even or odd way. Function $\widetilde{p}$ in each unit cube from $Q\setminus\Omega$ coincides with the shift of the function $\pm p$ to this cube.

At the same time, the extension $\widetilde{p}(\boldsymbol{x},t)$ satisfies in the domain Q for $t>0$ the Poisson equation

$$\triangle\,\widetilde{p} = \widetilde{f}. \tag{7.27}$$

At the boundary $S^0_0=\{x_2=\pm1\}\cup\{x_3=\pm1\}$ of the domain Q

$$\frac{\partial\widetilde{p}}{\partial n} = 0 \tag{7.28}$$

and at the boundary $S^i_0=\{x_1=(\pm1)^i\}$ Q

$$\widetilde{p} = 0. \tag{7.29}$$

Due to the boundary condition (7.29) and the inequality

$$\max_{0<t<T}\|\widetilde{f}(.,t)\|_{q,Q} \leqslant M(M_p),$$

for the solutions to the boundary – value problem (7.27) – (7.29) holds true the inner estimate (Lemma 9.12 §11, Chapter 9, [37])

$$\max_{0<t<T}\|\nabla\,\widetilde{p}(.,t)\|^{(1)}_{q,\Omega} \leqslant \max_{0<t<T}\|\widetilde{f}(.,t)\|_{q,Q} \leqslant M(q,M_p), \tag{7.30}$$

where $q>1$ is any.

Let $\gamma = \dfrac{q-3}{3}$. Then

$$|\nabla \widetilde{p}|^{(0)}_{\Omega_T} + \langle \nabla \widetilde{p}\rangle^{(\gamma)}_{x,\Omega_T} \leqslant \max_{0<t<T} \|\nabla \widetilde{p}(.,t)\|^{(1)}_{q,\Omega} \leqslant M(\gamma, M_p) \tag{7.31}$$

[25] (Theorem 2.1, §2, Chapter II), which proves the boundedness of the pressure gradient and its Helder norm with respect to the spatial variables.

To estimate the Helder norm of the pressure gradient with respect to time time, we consider the differences

$$D_h\widetilde{p}(\boldsymbol{x},t) = \frac{1}{h^{\frac{\gamma}{2}}}\big(\widetilde{p}(\boldsymbol{x},t+h) - \widetilde{p}(\boldsymbol{x},t)\big),$$

$$D_h\widetilde{f}(\boldsymbol{x},t) = \frac{1}{h^{\frac{\gamma}{2}}}\big(\widetilde{f}(\boldsymbol{x},t+h) - \widetilde{f}(\boldsymbol{x},t)\big).$$

These finite differences satisfy the Poisson equation (7.27)) in the domain Q and boundary conditions (7.28)) and (7.29) at the boundary Q.

Therefore, for $D_h\widetilde{p}$ the estimates (7.30) and (7.31) are fair:

$$|D_h\nabla\widetilde{p}(\boldsymbol{x},t)| \leqslant |D_h\nabla\widetilde{p}|^{(0)}_{\Omega_T} \leqslant M(q, M_p).$$

The limit as $h \to 0$ in the last inequality proves the estimate

$$\langle \nabla p\rangle^{(\frac{\gamma}{2})}_{t,\Omega_T} \leqslant M(q, M_p),$$

which is together with the estimate (7.31) and Darcy's law (7.23) complete the proof of the Lemma. □

Lemma 7.6. *In conditions of Theorem 4 the limiting function $c(\boldsymbol{x},t)$ belongs to the space $\mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$ and the estimates*

$$|c|^{(2+\gamma)}_{\Omega_T} \leqslant M(M_c) \tag{7.32}$$

and

$$0 \leqslant c(\boldsymbol{x},t) \leqslant c^*, \ \ (\boldsymbol{x},t) \in \Omega_T \tag{7.33}$$

are fair for her.

The estimate (7.33) follows from the estimate (5.5) after the limit as $\varepsilon \to 0$.

The estimate (7.32) follows from the estimate (7.33) and the local estimates for classical solutions to the equation (4.18) [25] (Theorem 10.1, §10, Chapter IV).

The existence of the classical solutions to the diffusion – convection problem $\mathbb{H}(r)$ follows from the estimate (7.32) and Theorem 5.2 5.3 [25] (§5, Chapter IV).

7.5 Properties of operators $\mathbb{F}^v(r)$, $\mathbb{F}^p(r)$ and $\mathbb{F}^c(r)$.

By construction, operators $\mathbb{F}^v(r)$, $\mathbb{F}^p(r)$ and $\mathbb{F}^c(r)$ act from the set $\mathfrak{M}$ into the spaces $\mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_T)$, $\mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{\Omega}_T)$ and $\mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$ accordingly.

Takes place the following

Lemma 7.7. *In conditions of the Theorem 4 the operators $\mathbb{F}^v(r)$, $\mathbb{F}^p(r)$ and $\mathbb{F}^c(r)$ are Lipschitz continuous*

$$|\mathbb{F}^v(r_1) - \mathbb{F}^v(r_2)|^{(\gamma)}_{\Omega_T} + |\mathbb{F}^p(r_1) - \mathbb{F}^p(r_2)|^{(\gamma)}_{\Omega_T} + \\ |\mathbb{F}^c(r_1) - \mathbb{F}^c(r_2)|^{(2+\gamma)}_{\Omega_T} \leqslant M(M_0)|r_1 - r_2|^{(2+\gamma)}_{\Omega_T}. \quad (7.34)$$

The proof of this statement is standard and expresses the well-known fact of continuous dependence of solutions to linear elliptic and parabolic equations on coefficients.

8 Proof of 5.

It is easy to see, that the operator $\mathbb{F}(r)$, defined by formula (4.27) satisfies the Lipschitz condition. Moreover, for some small time interval $(0, T_1)$ it is compressive and displays the set $\mathfrak{M}_T$ into itself.

In fact, let $R_i(\boldsymbol{x}, t) = \mathbb{F}(r_i) = \int_0^t c_i(\boldsymbol{x}, \tau)d\tau$, $i = 1, 2$, where $c_i = \mathbb{F}^c(r_i)$.

One has:

$$0 \leqslant \mathbb{F}(r)(\boldsymbol{x}, t) \leqslant T_1\, c^*, \ |\mathbb{F}(r)|^{(2+\gamma)}_{\Omega_T} \leqslant T_1\, M(M_0), \\ |\mathbb{F}(r_1) - \mathbb{F}(r_2)|^{(2+\gamma)}_{\Omega_T} \leqslant T_1\, M(M_0)\, |r_1 - r_2|^{(2+\gamma)}_{\Omega_T}.$$

That is, on the interval $(0, T_1)$, where

$$T_1 < \min\{\frac{1}{2\, c^*}, \frac{M(M_0)}{2}\}$$

the operator $\mathbb{F}(r)$ is compressive and displays the set $\mathfrak{M}_{T_1}$ into itself.

Banach 's Theorem (Theorem 1, §4,Chapter II, [20]) guarantees us the existence of the unique fixed point $r^*(\boldsymbol{x}, t)$ from the set $\mathfrak{M}_{T_1}$ and, that is, the validity of the Theorem 5 on the time interval $(0, T_1)$.

Next we put $r_1(\boldsymbol{x}) = r^*(\boldsymbol{x}, T_1)$ and consider the problem $\mathbb{B}^\varepsilon(r)$ on the interval (T_1, T), where instead of function $r_0(\boldsymbol{x})$ we will take the function $r_1(\boldsymbol{x})$, and instead of function $\bar{r}(\boldsymbol{x}, t)$ we will take the function $\bar{r}_1(\boldsymbol{x}, t) = \max\{0,\, r_1(\boldsymbol{x}) - r(\boldsymbol{x}, t)\}$.

Let also $\Omega_{(T_1,T)} = \Omega \times (T_1, T)$ and

$$\mathfrak{M}_{(T_1,T)} = \{r \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_{(T_1,T)}, \ \ r(\boldsymbol{x}, T_1) = 0, \ 0 \leqslant r(\boldsymbol{x}, t) \leqslant \frac{1}{2},$$
$$-\theta \leqslant \frac{\partial r_1}{\partial t}(\boldsymbol{x}, t) \leqslant 0, \ |r_1|^{(2+\gamma)}_{\Omega_{(T_1,T)}} \leqslant M_0\}.$$

After that, as before, we will define solutions $\{\boldsymbol{v}_1, \, p_1, \, c_1\}$ and $\{r_1^*, \, \boldsymbol{v}_1, \, p_1, \, c_1\}$ of the problems $\mathbb{H}(r)$ and $\mathbb{H}$ accordingly , at intervals (T_1, T) (T_1, T_2).

Repeating the process we will find the time intervals (T_k, T_{k+1}), $k = 1, 2, 3, ...$ and a function r^*, is equal to the function r_k^* on the intervals (T_k, T_{k+1}), $k = 1, \, 2, \, ...$

Obviously, these functions solve the problem $\mathbb{H}$ on the intervals $(0, T_k)$. If $\lim\limits_{k\to\infty} T_k = \infty$, then the theorem is proved.

If $\lim\limits_{k\to\infty} T_k = T^* < \infty$ and $r^*(\boldsymbol{x}, T^*)$ is nonzero on some open set of $\overline{\Omega}$, then by virtue of the obtained estimates of the solutions of the problem $\mathbb{H}$ we can calculate the limits of solutions as $t \to T^*$ and then will find the solution of the problem $\mathbb{H}$ on the interval $(T^*, T^* + \delta)$ with some small $\delta > 0$, which contradicts our assumption.

Thus, the process will be terminated only if $r^*(\boldsymbol{x}, T^*) = 0$ on the set Ω. For $t > T^*$ the liquid will completely fill the domain Ω and its motion will be described by the Stokes equations, which completes the proof of the theorem.

References

[1] N. Kalia, V. Balakotaiah, Effect of medium heterogeneities on reactive dissolution of carbonates, Chemical Engineering Science, vol. 64, 2009, pp. 376 – 390.

[2] C. E. Cohen, D. Ding, M. Quintard, B. Bazin, From pore scale to wellbore scale: Impact of geometry on wormhole growth in carbonate acidization, Chemical Engineering Science, vol. 63, 2008, pp. 3088 – 3099.

[3] M. K. R. Panga, M. Ziauddin, V. Balakotaiah, Two-scale continuum model for simulation of wormholes incarbonate acidization, A.I.Ch.E.Journal, vol. 51, 2005, pp. 3231 – 3248.

[4] R. Burridge, J. B. Keller, Poroelasticity equations derived from microstructure, J. Acoust. Soc. Am., vol. 70, issue 4, 1981, pp. 1140 – 1146.

[5] E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Lecture Notes in Phys., 127, Springer-Verlag, BerlinNew York, 1980.

[6] R. P. Gilbert, J. Z. Lin, Acoustic waves in shallow inhomogeneous oceans with a poro-elastic seabed, ZAMM, V., vol 79, issue 4, 1997, pp. 1 – 12.

[7] J. L. Ferrin, A. Mikelic, Homogenizing the acoustic properties of a porous matrix containing an incompressible inviscid fluids, Math. Meth. Appl. Sci. 26, 2003, pp. 831 – 859.

[8] T. Levy, Fluids in porous media and suspension, Homogenization techniques for composite media, Lecture Notes in Phys., vol 272, Springer, Berlin, 1987, pp. 63 – 119.

[9] J. Sanchez-Hubert, Asymptotic study of the macroscopic behavior of a solid-liquid mixture, Math. Methods Appl. Sci., vol 2, 1980,pp. 1 – 18.

[10] V. V. Jikov, S. M. Kozlov, O. A. Oleinik, Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, 1994.

[11] V. V. Zhikov, Homogenization of elasticity problems on singular structures, Izv. Math., vol 66, issue 2, 2002, pp. 299 – 365.

[12] S. E. Pastukhova, Homogenization of the stationary Stokes system in a perforated domain with a mixed condition on the boundary of cavities, Differ. Equ., vol 36, issue 5, 2000, 755 – 766.

[13] N. Bakhvalov, G. Panasenko, Homogenisation: averaging processes in periodic media, Mathematical problems in the mechanics of composite materials, Math. Appl. (Soviet Ser.), 36, Dordrecht, Kluwer Acad. Publ., 1989.

[14] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., vol 20, issue 3, 1989, pp. 608 – 623.

[15] A. Meirmanov, Mathematical models for poroelastic flow, Paris, Atlantis Press, 2014.

[16] L. V. Ovsaynnikov, Introduction to continuum mechanics, parts I, II, Novosibirsk, Novosibirsk State University, 1977 (russian).

[17] R. D. O'Dea at al, A multiscale analysis of nutrient transport and biological tissue growth in vitro, Mathematical Medicine and Biology https://doi.org/10.1093/imammb/dqu015 doi:10.1093/imammb/dqnxxx, 2015, pp. 1-23.

[18] A. Meirmanov, O. Galtsev, R. Zimin, Free boundaries in rock mechanics, Walter de Gruyter GmbH, Berlin-Boston, 2017.

[19] A. Meirmanov, The Stefan Problem, Berlin-New York, Walter de Gruyter, 199.

[20] A. N. Kolmogorov, S. V. Fomin, Introductory real analysis, Dover Publications, INC., New York, 1975.

[21] B. G. Galerkin, Rods and plates. Series in some questions of elastic equilibrium of rods and plates, Bulletin of Engineers, vol. 1, 1915, pp. 897908 (russian).

[22] O. A. Oleinik, A method of solution of the general Stefan problem, Doklady Mathematics, vo. 135, issue 5, 1960, 1054 – 1058.

[23] S. L. Kamenomostskaya, On Stefan's problem, Mathem. Sbornik, vol. 53(95), issue 4, 1961, 48951.

[24] A. Friedman, D. Kinderlehrer, A one phase Stefan problem, Indiana Univ. Math. J., vol. 24, 1975, pp. 10051035.

[25] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Uraltseva, Linear and quasilinear equations of parabolic type, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, R.I., 1968.

[26] J. L. Lions, Quelques metodes de resolution des problemes aux limites non lineaire, Dunon Gauthier – Villars, Paris, 1969.

[27] J. P. Aubin, Un theoreme de compacite, C. R. Acad. Sc., vol. 256, 1963, 5042 – 5044.

[28] A. Meirmanov, R. Zimin, Some compactness result for periodic structures and its application to the homogenization of a diffusion – convection equation, Electronic Journal of Differential Equations, vol. 2011, issue 115, 2011, pp. 1 – 11.

[29] R. A. Adams, Sobolev spaces, Academic Press, New York, 1975.

[30] V. P.Mikhailov, A. K. Guschin, Additional chapters of the course Equations of Mathematical Physics, Lecture courses NOC, issue 7, Steklov Mathematical Institute of Russian Academy of Science, Moscow 2007 (russian).

[31] C. Conca, On the application of the homogenization theory to a class of problems arising in fluid mechanics, J. Math. Pures Appl., vol. 64, issue 1, 1985,pp. 31 – 75.

[32] E. Acerbi, V. Chiad'o Piat, G. Dal Maso, D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Anal., vol. 18, issue 5, 1992, pp. 481 – 496.

[33] A. Yu. Goretski, S. N. Kruzhkov, G. A. Chechkin, Partial differential equations of the first order, Moscow State University, Moscow, 1999.

[34] D. Sansone, Equazioni differenziali nel campo real, parte prima, Zanichell, Bologna, 1948

[35] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flo, Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[36] V I. Smirnov, Higher mathematics course, vol. 4, Nauka, ., 1974 (russian).

[37] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer – Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[38] V. A. Il'in, E. G. Poznyak, Linear algebra, Nauka, . 1999.